Structure preserving model reduction of network systems

Jacquelien Scherpen

Jan C. Willems Center for Systems and Control
&
Engineering and Technology institute Groningen

Joint work with a.o. **Xiaodong Cheng**, Yu Kawano, Nima Monshizadeh, Arjan van der Schaft, Claudio De Persis, Hildeberto Jardón-Kojakhmetov, ...

25 May 2017, ACC Seattle

university of groningen

Outline

1. Introduction
2. Preliminaries
3. Clustering-based model reduction
   - Structure-preserving projection
   - Error bound
   - Properties of the reduced model
   - Some remarks
   - Small world example
4. Clustering-based reduction of second order network systems
5. Generalized balancing
6. Conclusion and outlook

university of groningen

1 Introduction

2 Preliminaries

3 Clustering-based model reduction
- Structure-preserving projection
- Error bound
- Properties of the reduced model
- Some remarks
- Small world example

4 Clustering-based reduction of second order network systems

5 Generalized balancing

6 Conclusion and outlook

university of groningen

Background

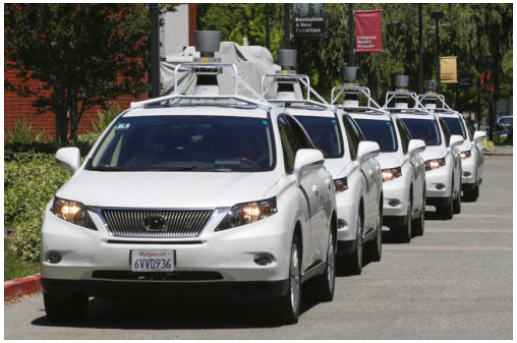
Network models capture the behaviors and dynamics of many interconnected physical systems.



Power grid



Robots in our lab



Transportation network

university of groningen

Background

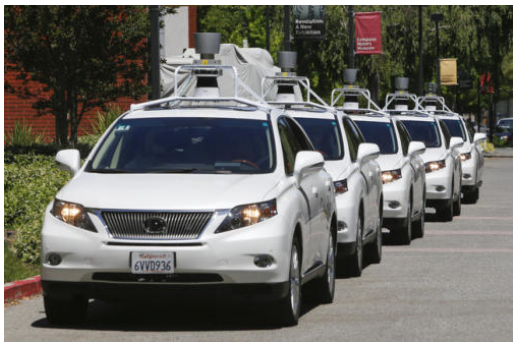
Network models capture the behaviors and dynamics of many interconnected physical systems.



Power grid

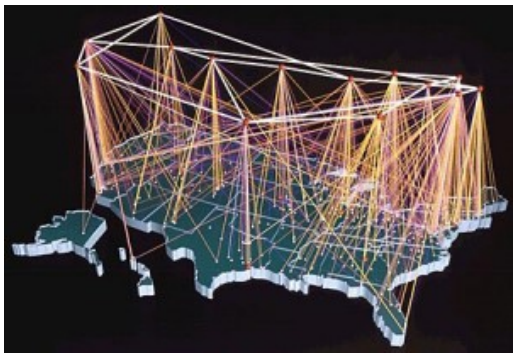


Robots in our lab



Transportation network

Network systems tend to be large-scale.







university of groningen

Background





Social networks, in organizations, social media,.... (from the next web)

university of groningen

Background

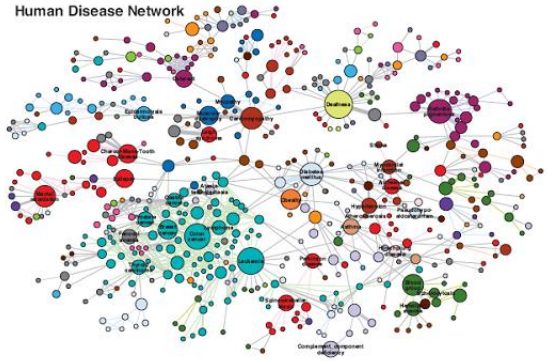




Social networks, in organizations, social media,.... (from the next web)



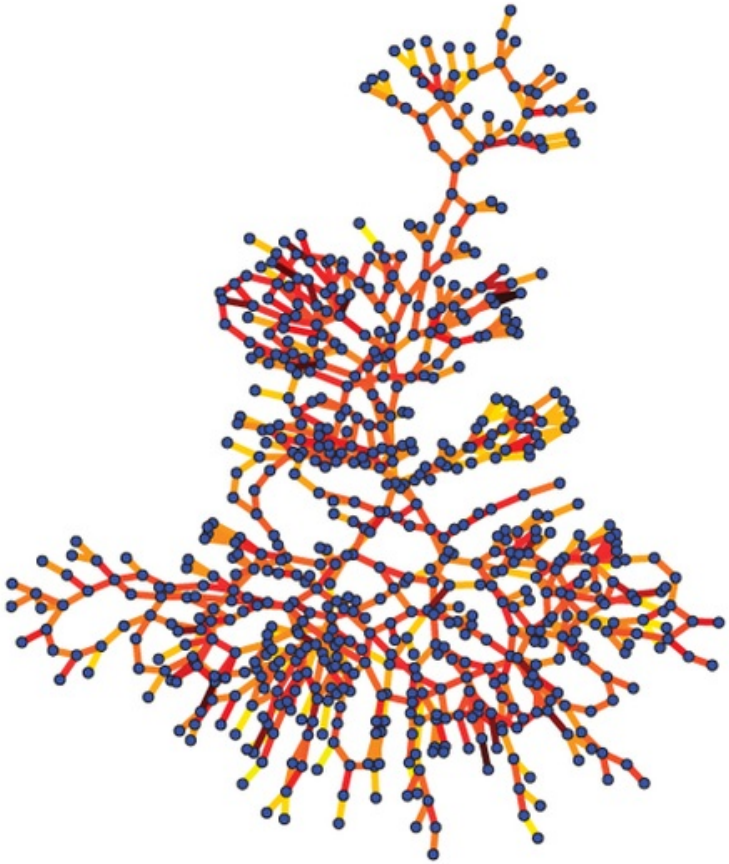
Biological networks (The protein interaction network of Treponema pallidum, Wikimedia commons)



Disease network (from CCSB)

university of groningen

Power grid of Northern Italy [1]



Representation of the physical network of transmission lines, which has 678 nodes and 822 edges.

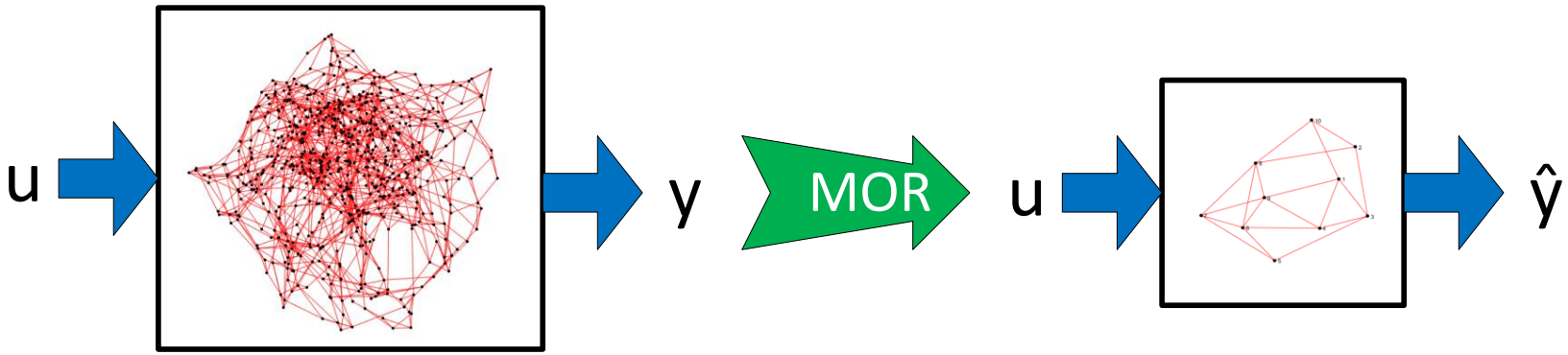
Such large-scale network complicates analysis and synthesis.

Apart from the network structure, the physical structure important, i.e., 2nd order structure.

[1] Motter, Adilson E., et al. "Spontaneous synchrony in power-grid networks." Nature Physics 9.3 (2013): 191-197.

university of groningen

Problem setting



Model reduction with preservation of network structure

Systems with network structure have useful properties.

- Consensus and synchronization.
- Spatial structure is crucial for distributed controller design, sensor allocation, etc..
- Specific physical structure, such as second order structure, is useful.
- Reduced-order models ease analysis, controller design, etc.

Order reduction methods

Various existing model order reduction methods:

- Balanced realizations based order reduction.

  Linear: Moore '81, and many more, overview Antoulas '05, Nonlinear: S. '93, Fujimoto/S'10, and a few more.....



university of groningen

Order reduction methods

Various existing model order reduction methods:

- Balanced realizations based order reduction.

  Linear: Moore '81, and many more, overview Antoulas '05, Nonlinear: S. '93, Fujimoto/S'10, and a few more.....

- Moment matching methods, mainly based on transfer functions, extension to state space forms.

  Linear: many references, overview Antoulas '05, Linear and nonlinear: Astolfi'10 and a few more

university of groningen

Order reduction methods

Various existing model order reduction methods:

- Balanced realizations based order reduction.

  Linear: Moore '81, and many more, overview Antoulas '05, Nonlinear: S. '93, Fujimoto/S'10, and a few more.....

- Moment matching methods, mainly based on transfer functions, extension to state space forms.

  Linear: many references, overview Antoulas '05, Linear and nonlinear: Astolfi'10 and a few more

- Singular perturbation methods, or slow fast system separation for control systems.

  Kokotovic/Khalil/O'Reily'86, and many more for hyperbolic systems, non-hyperbolic Jardon-Kojakhmetov/S.'16,17

university of groningen

Order reduction methods

Various existing model order reduction methods:

- Balanced realizations based order reduction.

  Linear: Moore '81, and many more, overview Antoulas '05, Nonlinear: S. '93, Fujimoto/S'10, and a few more.....

- Moment matching methods, mainly based on transfer functions, extension to state space forms.

  Linear: many references, overview Antoulas '05, Linear and nonlinear: Astolfi'10 and a few more

- Singular perturbation methods, or slow fast system separation for control systems.

  Kokotovic/Khalil/O'Reily'86, and many more for hyperbolic systems, non-hyperbolic Jardon-Kojakhmetov/S.'16,17

Just applying these methods to networks does not preserve the network structure.

university of groningen

Order reduction methods for graphs

Clustering based model order reduction:

- Recent interest, clustering of nodes.

  Ishizaki et al. '14, Monshizadeh et al.'14, Van der Schaft '14, Besselink et al."16



university of groningen

Order reduction methods for graphs

Clustering based model order reduction:

- Recent interest, clustering of nodes.

  Ishizaki et al. '14, Monshizadeh et al.'14, Van der Schaft '14, Besselink et al."16

Structure preservation for power systems reduction:

- Kron reduction.

  Van der Schaft'10, Dörfler/Bullo'13, Caliskan/Tabuada'14

- Projection based reduction.

  Monshizadeh et al.'17

university of groningen

Order reduction methods for graphs

Clustering based model order reduction:

- Recent interest, clustering of nodes.

  Ishizaki et al. '14, Monshizadeh et al.'14, Van der Schaft '14, Besselink et al."16

Structure preservation for power systems reduction:

- Kron reduction.

  Van der Schaft'10, Dörfler/Bullo'13, Caliskan/Tabuada'14

- Projection based reduction.

  Monshizadeh et al.'17

Various issues to deal with, topic of this talk!

university of groningen

1 Introduction

2 Preliminaries

3 Clustering-based model reduction
- Structure-preserving projection
- Error bound
- Properties of the reduced model
- Some remarks
- Small world example

4 Clustering-based reduction of second order network systems

5 Generalized balancing

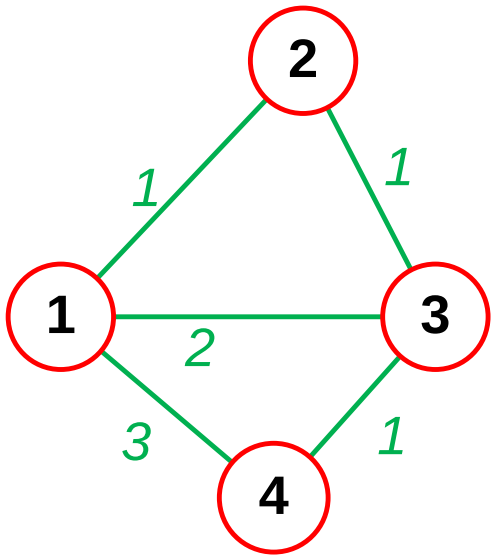
6 Conclusion and outlook

university of groningen

Graph theory

- An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $|\mathcal{V}| = n$, $|\mathcal{E}| = m$.
  $\mathcal{V}$: a set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$: a set of edges, $\mathcal{W}$: the weights of edges
- The Laplacian matrix of the undirected graph $\mathcal{G}$ is L, defined by

$$L_{ij} = \begin{cases} \text{degree of node } i, & i = j \\ -w_{ij}, & i \neq j \text{ and node } i \text{ is adjacent to } j \\ 0, & \text{otherwise} \end{cases}$$



Laplacian matrix

$$L = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 2 & -1 & 0 \\ -2 & -1 & 4 & -1 \\ -3 & 0 & -1 & 4 \end{bmatrix}.$$

university of groningen

Network system

Consider a network of single integrator systems:

$$\Sigma : M\dot{x} = -Lx + Bu.$$

- states: $x \in \mathbb{R}^n$, inputs: $u \in \mathbb{R}^p$
- evolves as a weighted connected undirected graph $\mathcal{G}$

university of groningen

Network system

Consider a network of single integrator systems:

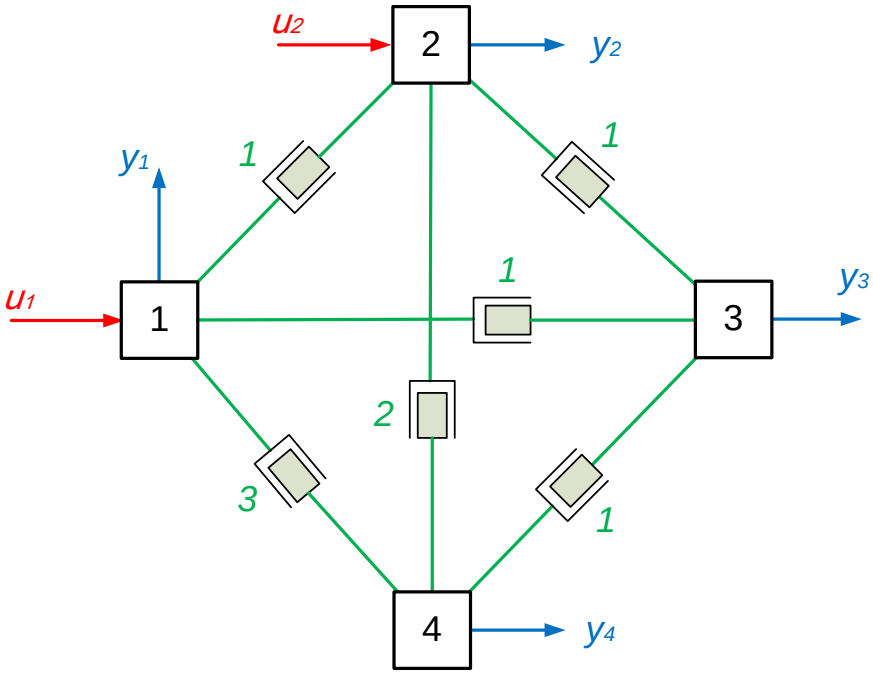
$$\Sigma : M\dot{x} = -Lx + Bu.$$

- states: $x \in \mathbb{R}^n$, inputs: $u \in \mathbb{R}^p$
- evolves as a weighted connected undirected graph $\mathcal{G}$

- **Specific structure**:
  M is diagonal positive definite.
  L is a Laplacian matrix of $\mathcal{G}$.
- Semistable, NOT asymptotically stable

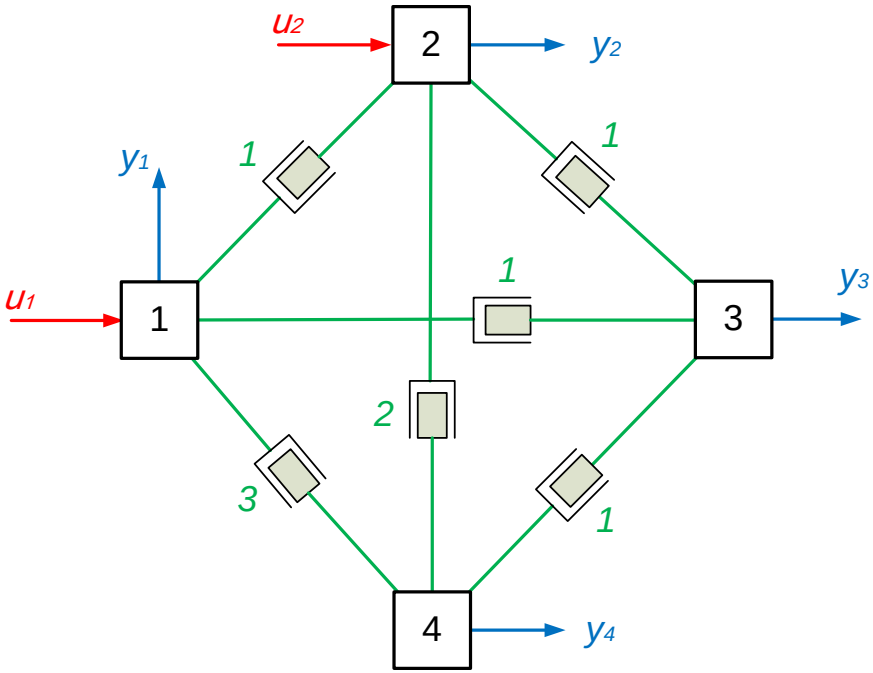
university of groningen

Example (mass-damper system)



university of groningen

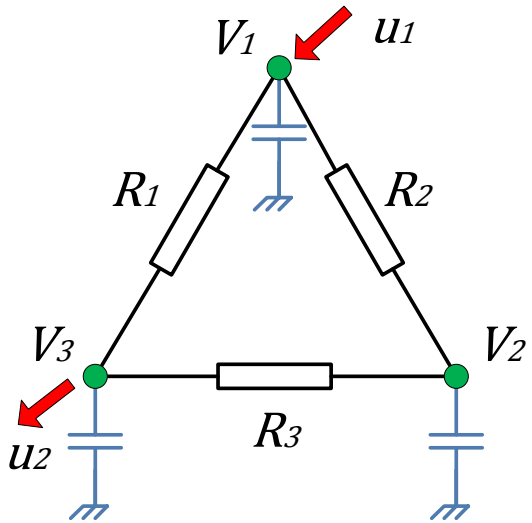
Example (mass-damper system)



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = - \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 4 & -1 \\ -3 & -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

university of groningen

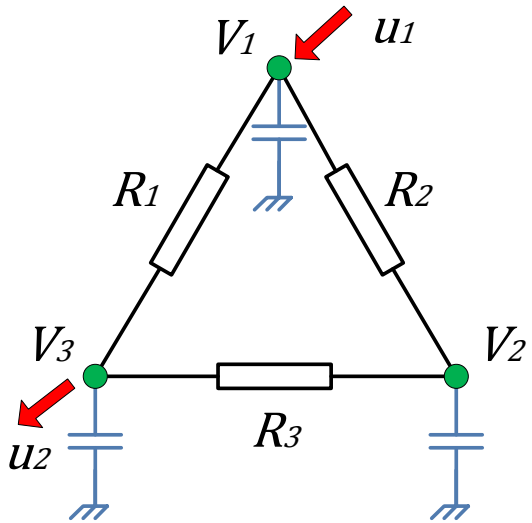
Example (electrical network)



- V_i: voltage of node i
- R_i: resistors
- u_i: current sources or sinks
- unit capacities

university of groningen

Example (electrical network)



- V_i: voltage of node i
- R_i: resistors
- u_i: current sources or sinks
- unit capacities

Simple network system (Kirchhoff's voltage law):

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = - \begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_1} & -\frac{1}{R_2} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ \frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ 0 \\ -u_2 \end{bmatrix}$$

university of groningen

Example (continued)

Simple Network System:

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = - \underbrace{\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_1} & -\frac{1}{R_2} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ \frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix}}_{\text{Laplacian matrix (Kirchhoff matrix)}} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Similar to mass-damper system with external forces.

university of groningen

Example (continued)

Simple Network System:

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = - \underbrace{\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_1} & -\frac{1}{R_2} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ \frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix}}_{\text{Laplacian matrix (Kirchhoff matrix)}} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Similar to mass-damper system with external forces.

Network system

$$\boldsymbol{\Sigma}: \begin{cases} \dot{x} = -Lx + Bu, \\ y = Cx. \end{cases} \quad \text{with } x \in \mathbb{R}^n$$

Notice that Laplacian matrix L represents an undirected network, which satisfies $\boxed{L \geq 0,\ L\mathbf{1} = 0}$.

university of groningen

More general network systems

Results of this talk hold for a more general class of network systems (beyond single integrators), i.e., the dynamics of each agent is described by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bv_i \\ \theta_i &= Cx_i \end{aligned}$$

$x_i \in \mathbb{R}^{\bar{n}}$, $v_i \in \mathbb{R}^{\bar{m}}$, $\theta_i \in \mathbb{R}^{\bar{m}}$, with diffusive coupling rule

$$m_i v_i = -\Sigma_{j=1, j\neq I}^{n} w_{ij}(\theta_i - \theta_j) + \Sigma_{j=1}^{p} f_{ij} u_j$$

$m_i > 0$, $u_j \in \mathbb{R}^{\bar{m}}$, $j \in \{1, \ldots, p\}$

university of groningen

More general network systems

Results of this talk hold for a more general class of network systems (beyond single integrators), i.e., the dynamics of each agent is described by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bv_i \\ \theta_i &= Cx_i \end{aligned}$$

$x_i \in \mathbb{R}^{\bar{n}}$, $v_i \in \mathbb{R}^{\bar{m}}$, $\theta_i \in \mathbb{R}^{\bar{m}}$, with diffusive coupling rule

$$m_i v_i = -\Sigma_{j=1, j\neq I}^{n} w_{ij}(\theta_i - \theta_j) + \Sigma_{j=1}^{p} f_{ij} u_j$$

$m_i > 0$, $u_j \in \mathbb{R}^{\bar{m}}$, $j \in \{1, \ldots, p\}$ results in network dynamics given by

$$(M \otimes I)\dot{x} = (M \otimes A - L \otimes BC)\, x + (F \otimes B)u$$

where the Laplacian L includes the w_{ij}, and $M = \text{diag}(m_1, \ldots, m_n) > 0$. Semistable, NOT asymptotically stable.

1 Introduction

2 Preliminaries

3 Clustering-based model reduction
- Structure-preserving projection
- Error bound
- Properties of the reduced model
- Some remarks
- Small world example

4 Clustering-based reduction of second order network systems

5 Generalized balancing

6 Conclusion and outlook

university of groningen

Problem Formulation

Given a network system

$$\Sigma : (M \otimes I)\dot{x} = (M \otimes A - L \otimes BC)\, x + (F \otimes B)u.$$

with A no poles in the RHP, and the network system fulfills the synchronization property, i.e., $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$.

university of groningen

Problem Formulation

Given a network system

$$\Sigma : (M \otimes I)\dot{x} = (M \otimes A - L \otimes BC)\, x + (F \otimes B)u.$$

with A no poles in the RHP, and the network system fulfills the synchronization property, i.e., $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$.

Apply Petro-Galerkin projection with $W, V \in \mathbb{R}^{n\times r}$ $(r \ll n)$ full column rank, such that:

- $\hat{M} := W^T M V$ and $\hat{L} := W^T L V$ are again an inertia and Laplacian matrix, resp. Van der Schaft'14, Monshizadeh/Van der Schaft'14
- The reduced-order model $\hat{\Sigma}$ has the same structure as above and $\|\Sigma - \hat{\Sigma}\|$ is small enough in terms of $\mathcal{H}_2$- or $\mathcal{H}_\infty$-norms.

university of groningen

Literature

- Monshizadeh et al. [1]

  Restrict to special clustering: Almost equitable partition

- Besselink et al. [2]

  Restrict to special topology: Tree graph

- Ishizaki et al. [3]

  Loss of the structure of Laplacian matrix

---

[1] N. Monshizadeh, H. L. Trentelman, and M. K. Camlibel, Projection- Based Model Reduction of Multi-Agent Systems Using Graph Partitions, IEEE Trans. Contr. Net. Syst., vol. 1, pp. 145-154, Jun. 2014.
A.J. van der Schaft, On model reduction of physical network systems, pp. 1419-1425 in Proc. MTNS2014, Groningen, the Netherlands, July 2014.
N. Monshizadeh, A.J. van der Schaft, Structure-preserving model reduction of physical network systems by clustering, Proc. 53rd IEEE CDC, Los Angeles, CA, USA, Dec. 2014.

[2] B. Besselink; H. Sandberg; K. Johansson, "Clustering-Based Model Reduction of Networked Passive Systems," in IEEE TAC, vol.61, no.10, 2016

[3] T. Ishizaki, K. Kashima, J. I. Imura, and K. Aihara, Model reduction and clusterization of large-scale bidirectional networks, IEEE TAC, vol. 59, pp. 48-63, 2014.

university of groningen

Clustering of graphs

Definition (Clustering)

node set $\mathcal{V} \Longrightarrow$ nonempty disjoint subsets $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$.

university of groningen

Clustering of graphs

Definition (Clustering)

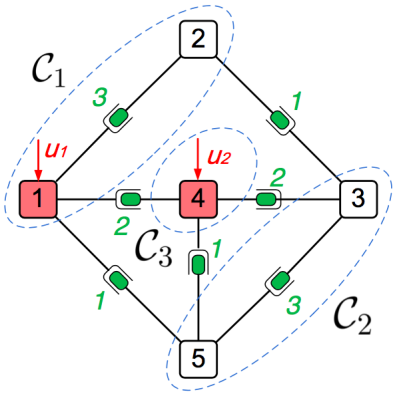
node set $\mathcal{V} \Longrightarrow$ nonempty disjoint subsets $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$.

Definition (Cluster matrix)

Consider a graph clustering $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$ of $\mathcal{G}$ with n nodes. **Cluster matrix** $P \in \mathbb{R}^{n \times r}$:

$$P := [p(\mathcal{C}_1), p(\mathcal{C}_2), \cdots, p(\mathcal{C}_r)],$$

$p(\mathcal{C}_i) \in \mathbb{R}^n$: the k-th element is 1 if $k \in \mathcal{C}_i$, 0 otherwise.



university of groningen

Clustering of graphs

Definition (Clustering)

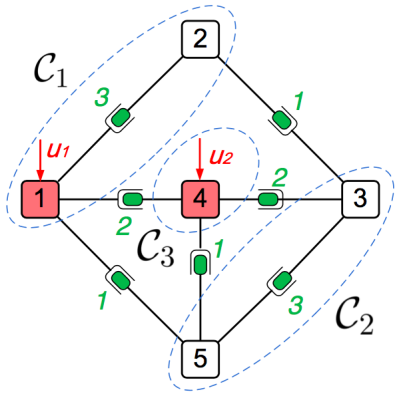
node set $\mathcal{V} \Longrightarrow$ nonempty disjoint subsets $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$.

Definition (Cluster matrix)

Consider a graph clustering $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$ of $\mathcal{G}$ with n nodes. **Cluster matrix** $P \in \mathbb{R}^{n \times r}$:

$$P := [p(\mathcal{C}_1), p(\mathcal{C}_2), \cdots, p(\mathcal{C}_r)],$$

$p(\mathcal{C}_i) \in \mathbb{R}^n$: the k-th element is 1 if $k \in \mathcal{C}_i$, 0 otherwise.

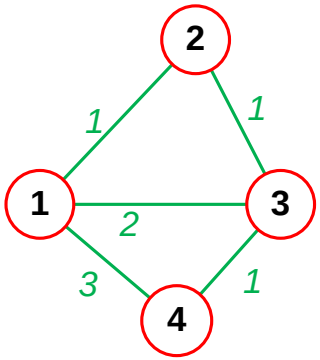


Take the cluster matrix for projection, i.e., $W = V = P$, then

$\hat{M} := P^T M P$ diagonal positive-definite;
$\hat{L} := P^T L P$ Laplacian of a connected undirected graph.

university of groningen

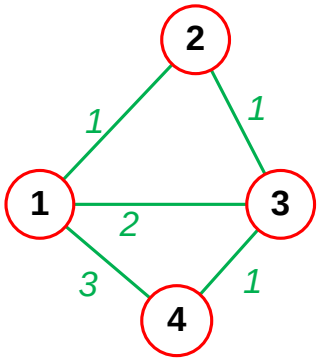
Example (structure preservation)



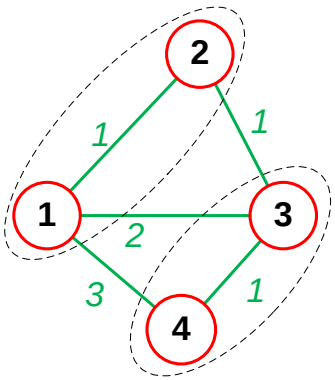
$$L = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 2 & -1 & 0 \\ -2 & -1 & 4 & -1 \\ -3 & 0 & -1 & 4 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

university of groningen

Example (structure preservation)



$$L = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 2 & -1 & 0 \\ -2 & -1 & 4 & -1 \\ -3 & 0 & -1 & 4 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$



$$P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$



$$\hat{L} = P^T L P = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}, \quad \hat{M} = P^T M P = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

university of groningen

Cluster selection

How to cluster, i.e.,

Problem

Find a clustering, with cluster matrix P such that the reduced-order model $\hat{\Sigma}$ approximates the original system Σ well.

university of groningen

Cluster selection

How to cluster, i.e.,

Problem

Find a clustering, with cluster matrix P such that the reduced-order model $\hat{\Sigma}$ approximates the original system Σ well.

Aggregate nodes with similar behavior:

- How to capture behavior?
- How to quantify the differences in behavior?

university of groningen

Cluster selection

How to cluster, i.e.,

Problem

Find a clustering, with cluster matrix P such that the reduced-order model $\hat{\Sigma}$ approximates the original system Σ well.

Aggregate nodes with similar behavior:

- How to capture behavior?
- How to quantify the differences in behavior?

Here we use

- input to state transfer functions of the nodes.
- and $\mathcal{H}_2$-norms for the difference.

university of groningen

Cluster selection

Consider the single integrator network system

$$M\dot{x} = -Lx + Bu.$$

- The i-th input-state transfer function

$$\eta_i(s) = e_i^T (sM + L)^{-1} B$$

Similar, but more involved expressions for the more general form of the network system.

university of groningen

Cluster selection

Consider the single integrator network system

$$M\dot{x} = -Lx + Bu.$$

- The i-th input-state transfer function

$$\eta_i(s) = e_i^T (sM + L)^{-1} B$$

- Difference of transfer functions

$$\eta_i(s) - \eta_j(s) = (e_i^T - e_j^T)(sM + L)^{-1} B$$

$\|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}$ quantifies the similarity of node i and j.

Similar, but more involved expressions for the more general form of the network system.

university of groningen

Boundedness

For general networks, under the assumptions on the A matrix and the synchronization property of the network, we have

$$\|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}^2 = \int_0^\infty \mathbf{tr}\left[(e_i^T \otimes I)\xi(t)\xi^T(t)(e_j \otimes I)\right] dt,$$

with $\xi(t)$ the state impulse response of the system. It can be proven to be always bounded!



university of groningen

Boundedness

For general networks, under the assumptions on the A matrix and the synchronization property of the network, we have

$$\|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}^2 = \int_0^\infty \mathbf{tr}\left[(e_i^T \otimes I)\xi(t)\xi^T(t)(e_j \otimes I)\right] dt,$$

with $\xi(t)$ the state impulse response of the system. It can be proven to be always bounded!

Note that this quantifies the (dis)similarity between node i and j in $\mathcal{H}_2$ sense.

university of groningen

Boundedness

For general networks, under the assumptions on the A matrix and the synchronization property of the network, we have

$$\|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}^2 = \int_0^\infty \mathbf{tr}\left[(e_i^T \otimes I)\xi(t)\xi^T(t)(e_j \otimes I)\right] dt,$$

with $\xi(t)$ the state impulse response of the system. It can be proven to be always bounded!

Note that this quantifies the (dis)similarity between node i and j in $\mathcal{H}_2$ sense.

(Recall: for a single integrator system:
$\|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} = \|(e_i^T - e_j^T)(sM + L)^{-1}B\|_{\mathcal{H}_2}$)

university of groningen

Recursive clustering algorithm

Definition (Dissimilarity Matrix $\mathcal{D} \in \mathbb{R}^{n \times n}$)

$$\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}$$



university of groningen

Recursive clustering algorithm

Definition (Dissimilarity Matrix $\mathcal{D} \in \mathbb{R}^{n \times n}$)

$$\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}$$

Algorithm (recursive clustering)

- Step 1: Compute $\mathcal{D}$ matrix
- Step 2: Find the minimal off-diagonal entry $\mathcal{D}_{\mu\nu}$.
- Step 3: Cluster nodes μ and ν
- Step 4: Generate $\hat{\Sigma}$ by P resulting from the clustering in Step 3.
- Step 5: Repeat Step 1 to Step 4 until obtaining r-th order model.

Other clusterings (e.g., hierarchical) possible.

university of groningen

Illustrative example

mass-damper network system:

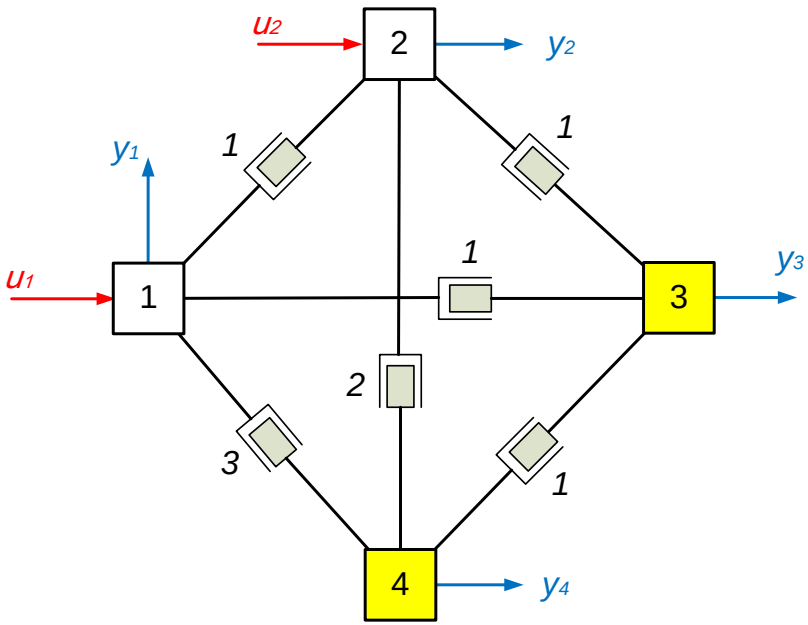


Figure: Topology of the original network

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

$$L = \begin{bmatrix} 6 & -1 & -2 & -3 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 4 & -1 \\ -3 & -1 & -1 & 5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T, \; C = I_4.$$

university of groningen

Illustrative example (continued)

- Step 1: Compute the dissimilarity matrix

$$\mathcal{D} = \begin{bmatrix} 0 & 0.2940 & 0.1342 & 0.1472 \\ 0.2940 & 0 & 0.2098 & 0.2322 \\ 0.1342 & 0.2098 & 0 & 0.0277 \\ 0.1472 & 0.2322 & 0.0277 & 0 \end{bmatrix}.$$

- Step 2: The minimal value is 0.0277, which implies that the nodes 3 and 4 have most similar behaviors.

university of groningen

Illustrative example (continued)

- Step 1: Compute the dissimilarity matrix

$$\mathcal{D} = \begin{bmatrix} 0 & 0.2940 & 0.1342 & 0.1472 \\ 0.2940 & 0 & 0.2098 & 0.2322 \\ 0.1342 & 0.2098 & 0 & 0.0277 \\ 0.1472 & 0.2322 & 0.0277 & 0 \end{bmatrix}.$$

- Step 2: The minimal value is 0.0277, which implies that the nodes 3 and 4 have most similar behaviors.
- Step 3: Clustering: $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{2\}$, $\mathcal{C}_3 = \{3, 4\}$.
- Step 4: Projection matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

university of groningen

Illustrative example (continued)

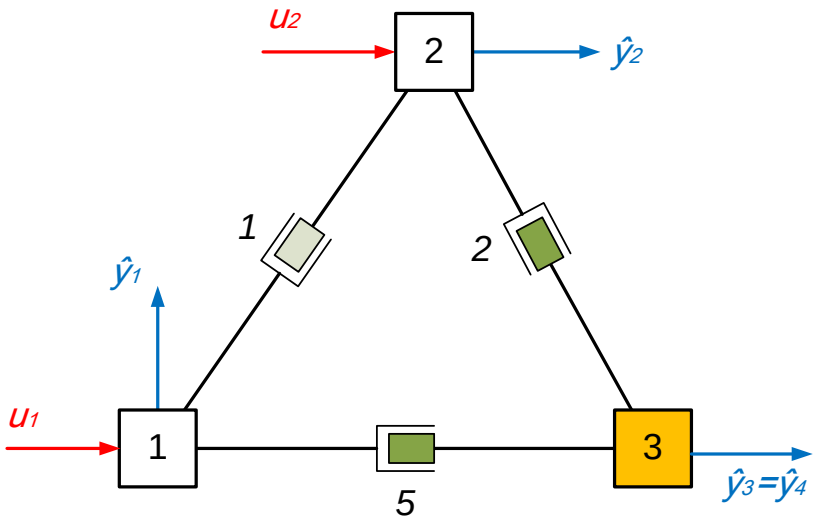


Figure: Topology of the reduced-order network

$$\hat{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

$$\hat{L} = \begin{bmatrix} 6 & -1 & -5 \\ -1 & 3 & -2 \\ -5 & -2 & 7 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T,$$

$$\hat{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

university of groningen

Error bound

- An error bound expression for one step clustering in the single integrator case is easily computable.



university of groningen

Error bound

- An error bound expression for one step clustering in the single integrator case is easily computable.
- An error bound for the general network description reduced to r clusters is given by dissimilarity between clusters, i.e.,

$$\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2} \leq \gamma_a \sum_{k=1}^{r} \max_{i,j \in \mathcal{C}_k} \mathcal{D}_{ij},$$

where γ_a is a scalar satisfying an LMI, i.e.,

$$\begin{bmatrix} M \otimes (A^T + A) - L \otimes (C^T B^T + BC) & L \otimes BC & -I \\ L \otimes C^T B^T & -\gamma_a I & I \\ -I & I & -\gamma_a I \end{bmatrix} \prec 0$$

university of groningen

Illustrative example (continued)

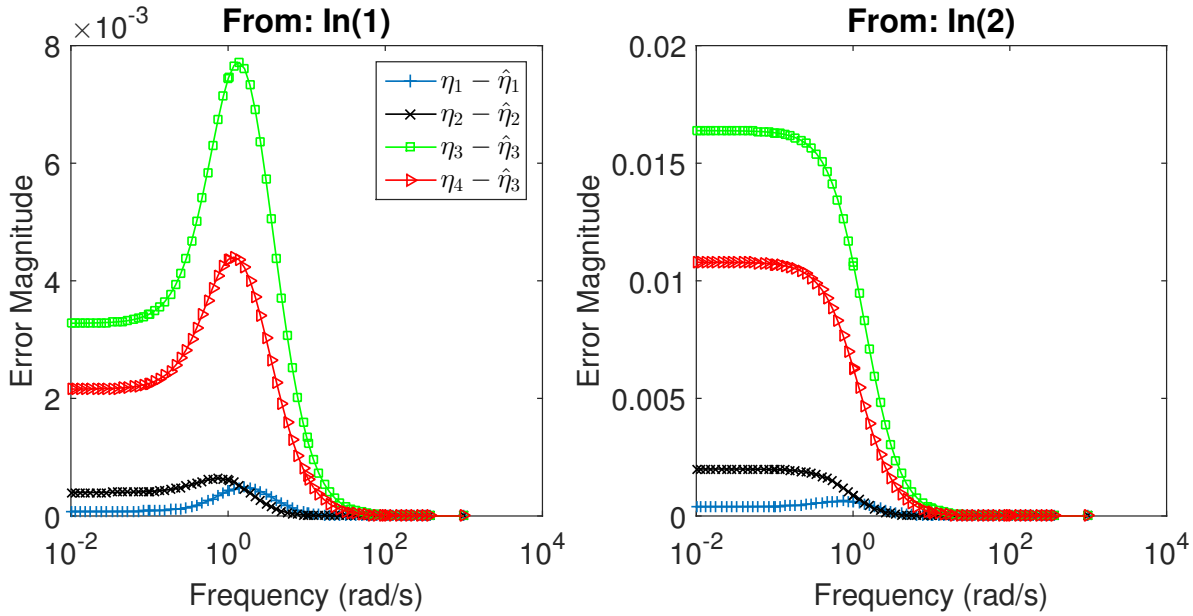


Figure: Plot of the error magnitude.

The error bounds are:

$$\|\eta_1(s) - \hat{\eta}_1(s)\| \leq 0.196, \ \|\eta_2(s) - \hat{\eta}_2(s)\| \leq 0.039,$$
$$\|\eta_3(s) - \hat{\eta}_3(s)\| \leq 0.046, \ \|\eta_4(s) - \hat{\eta}_3(s)\| \leq 0.037.$$

university of groningen

Properties of the reduced model

The reduced-order network system $\hat{\Sigma}$ has the following properties:

- The synchronization property is preserved, i.e., if the initial conditions are "equal" for the full order and reduced order network, then the nodes of the reduced order network converge to the same value as the nodes of the original network.



university of groningen

Properties of the reduced model

The reduced-order network system $\hat{\Sigma}$ has the following properties:

- The synchronization property is preserved, i.e., if the initial conditions are "equal" for the full order and reduced order network, then the nodes of the reduced order network converge to the same value as the nodes of the original network.

- $\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_2}$ and $\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_\infty}$ are bounded.



university of groningen

Properties of the reduced model

The reduced-order network system $\hat{\Sigma}$ has the following properties:

- The synchronization property is preserved, i.e., if the initial conditions are "equal" for the full order and reduced order network, then the nodes of the reduced order network converge to the same value as the nodes of the original network.
- $\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_2}$ and $\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_\infty}$ are bounded.
- If nodes μ and ν are clustered
  Then, $\eta_i(s) - \hat{\eta}_i(s) = K_i(s)[\eta_\mu(s) - \eta_\nu(s)]$

university of groningen

Minimal network realization

What is a minimal network realization?

Definition

A network realization is minimal if there are no 0-dissimilar vertices in the network, i.e., if $\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} \neq 0$ for $i \neq j$.

Thus, if $D_{ij} = 0$, then node i and j are 0-dissimilar, and the network is non-minimal.

university of groningen

Minimal network realization

What is a minimal network realization?

Definition

A network realization is minimal if there are no 0-dissimilar vertices in the network, i.e., if $\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} \neq 0$ for $i \neq j$.

Thus, if $D_{ij} = 0$, then node i and j are 0-dissimilar, and the network is non-minimal.

Relation with "conventional" minimality of the overall multi-agent system:

Relation

If Σ is minimal (controllable and observable), then Σ is also a minimal network realization.

Not vice versa!

university of groningen

Computational issues

Normally for calculating the $\mathcal{H}_2$ norm (necessary for the dissimilarity) the controllability Gramian can be used. However, the system is not asymptotically stable.

- Solution: Use a type of network controllability Gramian.



Computational issues

Normally for calculating the $\mathcal{H}_2$ norm (necessary for the dissimilarity) the controllability Gramian can be used. However, the system is not asymptotically stable.

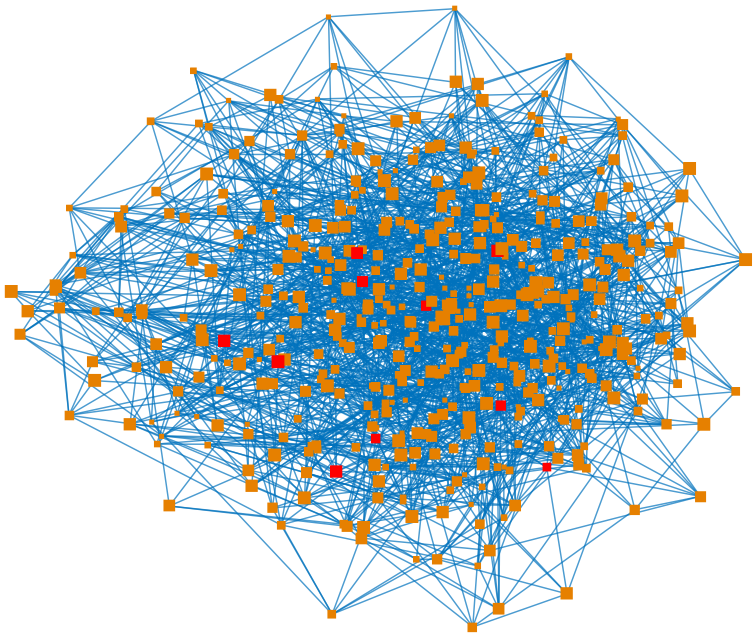
- Solution: Use a type of network controllability Gramian.

Iterative clustering and hierarchical clustering can be computationally expensive.

- Solution: Hierarchical clustering considers proximity of various clusters, and can be approximated by taking average dissimilarities (per cluster) of vertices to determine the proximity.

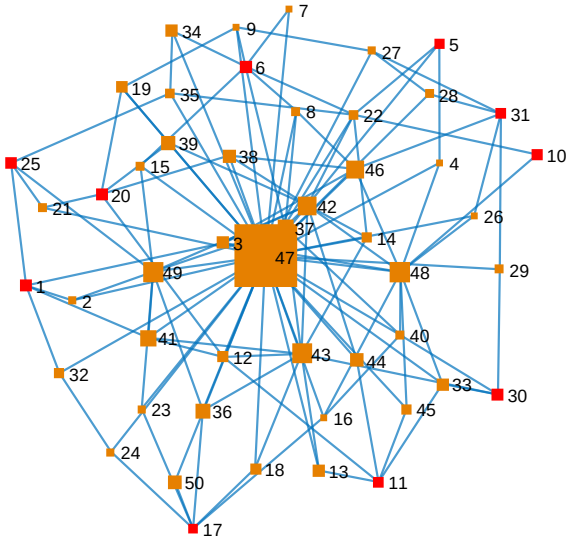
university of groningen

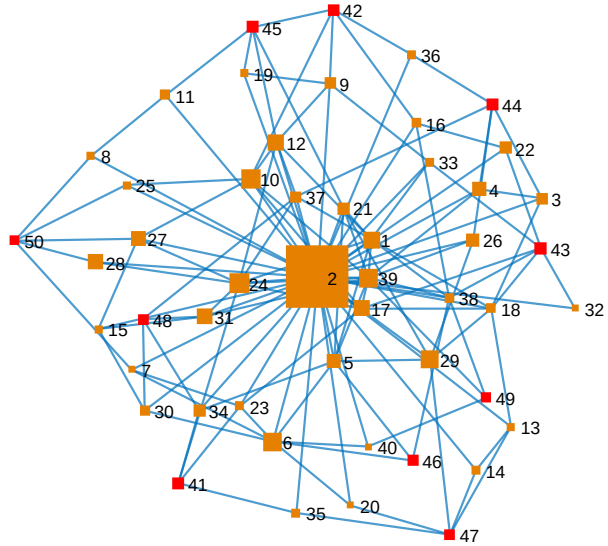
Small world network example



Left: Watts-Strogatz network with 500 nodes, 2000 edges, and 10 inputs.

Bottom: Approximated by iterative clustering (left) and hierarchical clustering (right) with 50 clusters.



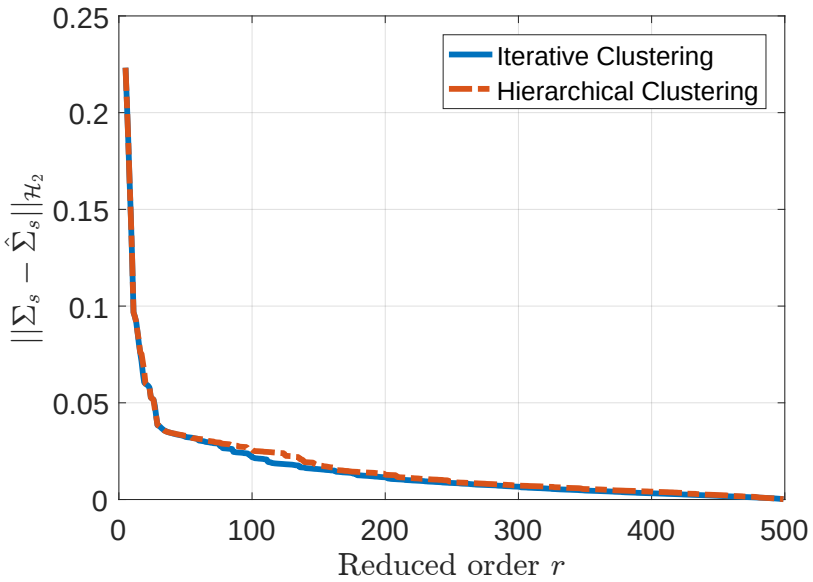


university of groningen

Small world network example continued

Table: Comparison of two algorithms for a reduced order of 50

Algorithm	$\|\Sigma_s - \hat{\Sigma}_s\|_{\mathcal{H}_2}$	Total computation time
Iterative clustering	0.0324	4142.5729s
(Appr.) hierarchical clustering	0.0330	38.5899s



Approximation error versus reduced order r.

university of groningen

1 Introduction

2 Preliminaries

3 Clustering-based model reduction
- Structure-preserving projection
- Error bound
- Properties of the reduced model
- Some remarks
- Small world example

4 Clustering-based reduction of second order network systems

5 Generalized balancing

6 Conclusion and outlook

university of groningen

Second-order network systems

Mass-spring-damper network

$$M\ddot{x} + D\dot{x} + Lx = Fu. \tag{1}$$

Example: linearized coupled oscillator model in the power grid.

university of groningen

Second-order network systems

Mass-spring-damper network

$$M\ddot{x} + D\dot{x} + Lx = Fu. \tag{1}$$

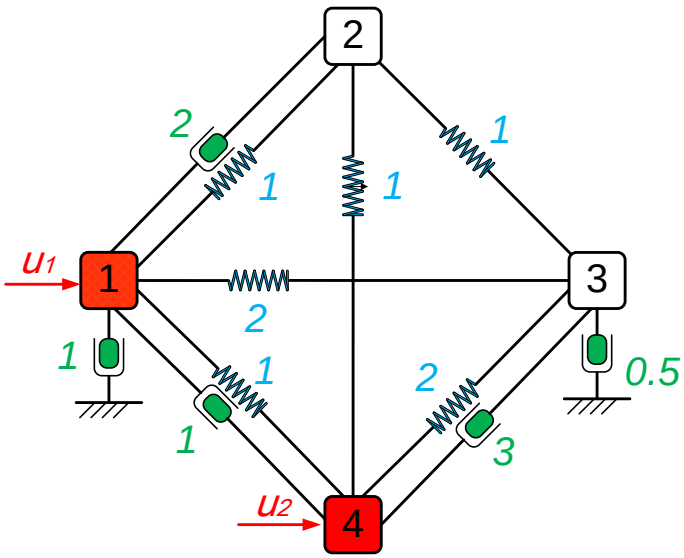
Example: linearized coupled oscillator model in the power grid.

Structural conditions

- M: diagonal positive definite;
- D: positive definite;
- L: Laplacian matrix of the weighted undirected graph describing the coupling among nodes.

university of groningen

A second-order network example



$$L = \begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 5 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix},$$

$$D = \begin{bmatrix} 4 & -2 & 0 & -1 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 3.5 & -3 \\ -1 & 0 & -3 & 4 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

university of groningen

Problem formulation

- Given a second-order network system

$$\boldsymbol{\Sigma} : M\ddot{x} + D\dot{x} + Lx = Fu, \;\; y = x.$$

- Apply Petrov-Galerkin projection with the cluster matrix, i.e., $P \in \mathbb{R}^{n \times r}$ $(r \ll n)$

university of groningen

Problem formulation

- Given a second-order network system

$$\boldsymbol{\Sigma} : M\ddot{x} + D\dot{x} + Lx = Fu, \;\; y = x.$$

- Apply Petrov-Galerkin projection with the cluster matrix, i.e., $P \in \mathbb{R}^{n \times r}$ $(r \ll n)$
- The reduced-order model

$$\hat{\boldsymbol{\Sigma}} : \hat{M}\ddot{z} + \hat{D}\dot{z} + \hat{L}z = P^T F u,$$

with $\hat{M} := P^T M P$, $\hat{D} := P^T D P$, $\hat{L} := P^T L P$.

university of groningen

Problem formulation

- Given a second-order network system

$$\Sigma : M\ddot{x} + D\dot{x} + Lx = Fu, \;\; y = x.$$

- Apply Petrov-Galerkin projection with the cluster matrix, i.e., $P \in \mathbb{R}^{n \times r}$ $(r \ll n)$
- The reduced-order model

$$\hat{\Sigma} : \hat{M}\ddot{z} + \hat{D}\dot{z} + \hat{L}z = P^T F u,$$

with $\hat{M} := P^T M P$, $\hat{D} := P^T D P$, $\hat{L} := P^T L P$.

Find a suitable clustering such that $\|\Sigma - \hat{\Sigma}\|$ is small enough in terms of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norms



university of groningen

Existing methods

"Conventional" second-order model reduction methods

- Second-order balanced truncation

  Meyer & Srinivasan'96, Reis & Stykel, Benner & Saak, Losse & Mehrmann, Y. Chahlaoui, et al.'06. .....

- Krylov subspace methods

  Salimbahrami & Lohmann, Bai & Su'05, Su & Craig Jr. ...

Limitation: network structure is not preserved.



university of groningen

Existing methods

"Conventional" second-order model reduction methods

- Second-order balanced truncation

  Meyer & Srinivasan'96, Reis & Stykel, Benner & Saak, Losse & Mehrmann, Y. Chahlaoui, et al.'06. .....

- Krylov subspace methods

  Salimbahrami & Lohmann, Bai & Su'05, Su & Craig Jr. ...

Limitation: network structure is not preserved.

Clustering-based approach

- Ishizaki & Imura '15

Limitation:

- Loss of the structure of Laplacian matrix
- The system has to be asymptotically stable

university of groningen

Network Gramian

Again: dissimilarity matrix defined and relevant for clustering!
For computation, the network Gramian is relevant, i.e.,

Network controllability Gramian

$$\mathcal{P} = \int_0^\infty (e^{\mathcal{A}\tau} - \mathcal{J})\mathcal{B}\mathcal{B}^T(e^{\mathcal{A}^T\tau} - \mathcal{J}^T)d\tau.$$

where $\mathcal{A} = \begin{bmatrix} \mathbf{0} & I \\ -M^{-1}L & -M^{-1}D \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} \mathbf{0} \\ M^{-1}F \end{bmatrix}$, and $\mathcal{J} = \lim_{\tau\to\infty} e^{\mathcal{A}\tau} = \frac{1}{\mathbf{1}^T D\mathbf{1}} \begin{bmatrix} \mathbf{1}\mathbf{1}^T D & \mathbf{1}\mathbf{1}^T M \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$.

university of groningen

Network Gramian

Again: dissimilarity matrix defined and relevant for clustering!
For computation, the network Gramian is relevant, i.e.,

Network controllability Gramian

$$\mathcal{P} = \int_0^\infty (e^{\mathcal{A}\tau} - \mathcal{J})\mathcal{B}\mathcal{B}^T(e^{\mathcal{A}^T\tau} - \mathcal{J}^T)d\tau.$$

where $\mathcal{A} = \begin{bmatrix} \mathbf{0} & I \\ -M^{-1}L & -M^{-1}D \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} \mathbf{0} \\ M^{-1}F \end{bmatrix}$, and $\mathcal{J} = \lim_{\tau\to\infty} e^{\mathcal{A}\tau} = \frac{1}{\mathbf{1}^T D\mathbf{1}} \begin{bmatrix} \mathbf{1}\mathbf{1}^T D & \mathbf{1}\mathbf{1}^T M \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$.

$\mathcal{P}$ fulfills a Lyapunov type of equation, and the rank has a direct relation with controllability of the original semi-stable system!

university of groningen

Network Gramian

The network Gramian $\mathcal{P}$ is useful for fast computation of dissimilarity and error bound computation, i.e., for dissimilarity:

$$\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} = \sqrt{H\mathcal{P}H^T}, \text{ with } H = [(e_i - e_j)^T, \mathbf{0}_{1\times n}]$$

university of groningen

Network Gramian

The network Gramian $\mathcal{P}$ is useful for fast computation of dissimilarity and error bound computation, i.e., for dissimilarity:

$$\mathcal{D}_{ij} = \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2} = \sqrt{H\mathcal{P}H^T}, \text{ with } H = [(e_i - e_j)^T, \mathbf{0}_{1\times n}]$$

Error bound

$$\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_2} = \sqrt{\mathbf{tr}\left(\mathcal{C}_\mathbf{e} \begin{bmatrix} \mathcal{P}_n & \mathcal{P}_\times \\ \mathcal{P}_\times^T & \mathcal{P}_r \end{bmatrix} \mathcal{C}_\mathbf{e}^T\right)},$$

$\mathcal{P}_n$ is the network controllability Gramian of the original network
$\mathcal{P}_r$ is of the reduced order network
$\mathcal{P}_\times$ is a coupling term that is a solution to a Sylvester like equation.
$\mathcal{C}_\mathbf{e}$ is the output matrix of the error system.

university of groningen

Illustrative example

Mass-damper-spring system:

$$\mathcal{D} = \begin{bmatrix} 0 & 0.1920 & 0.2449 & 0.2424 \\ 0.1920 & 0 & 0.0940 & 0.1163 \\ 0.2449 & 0.0940 & 0 & 0.0333 \\ 0.2424 & 0.1163 & 0.0333 & 0 \end{bmatrix}.$$

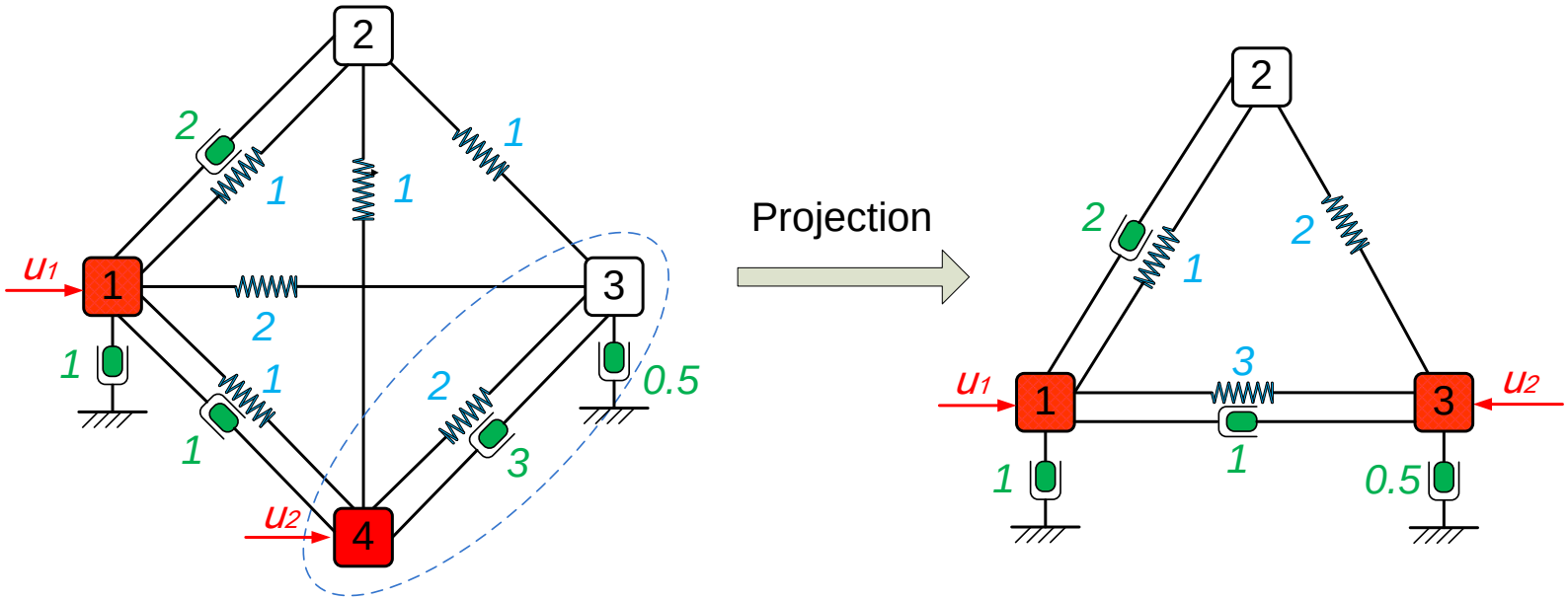


Figure: Illustration of second-order network system

Illustrative example (continued)

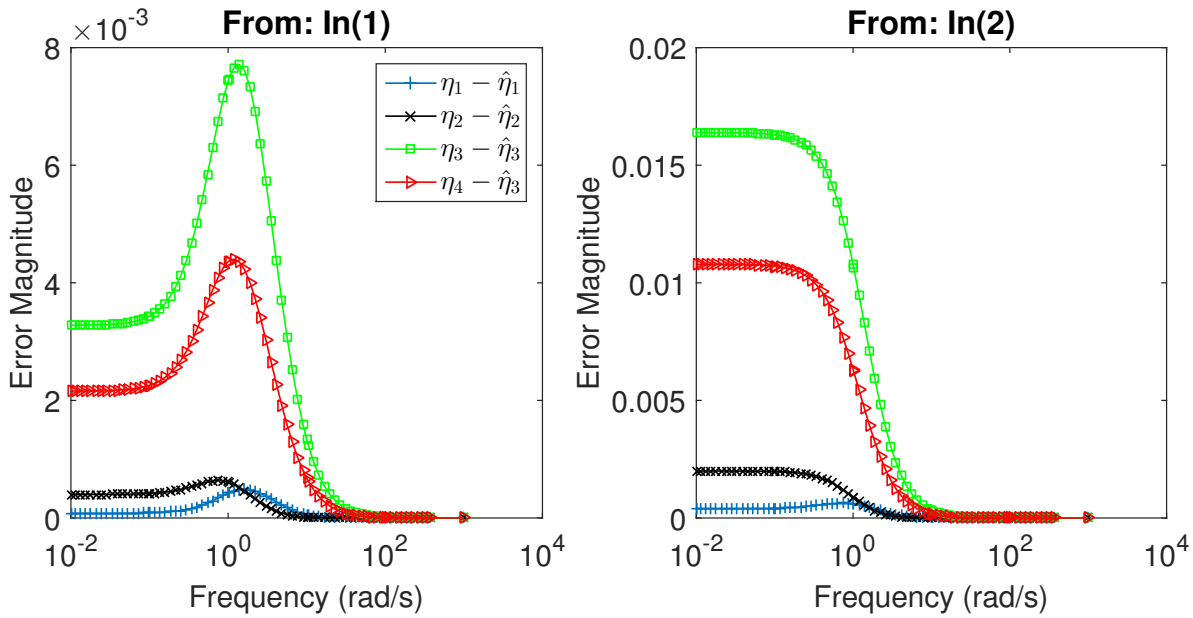


Figure: Plot of the error magnitude.

The error bounds are:

$$\|\eta_1(s) - \hat{\eta}_1(s)\| \leq 0.196, \ \|\eta_2(s) - \hat{\eta}_2(s)\| \leq 0.039,$$
$$\|\eta_3(s) - \hat{\eta}_3(s)\| \leq 0.046, \ \|\eta_4(s) - \hat{\eta}_3(s)\| \leq 0.037.$$

university of groningen

Some remarks

- Small world second order network example with various clustering methods, and how to use these methods can be found in our TAC'17 paper Cheng/Kawano/S.'17.



university of groningen

Some remarks

- Small world second order network example with various clustering methods, and how to use these methods can be found in our TAC'17 paper Cheng/Kawano/S.'17.
- Applicable for second order equations description of the power grid.

university of groningen

Some remarks

- Small world second order network example with various clustering methods, and how to use these methods can be found in our TAC'17 paper Cheng/Kawano/S.'17.
- Applicable for second order equations description of the power grid.
- In another TAC'17 paper we develop a projection for exact model reduction for electrical networks with power loads based on a projected incidence matrix which provides a new decomposition of the Laplacian. Monshizadeh et al.'17.

university of groningen

Some remarks

- Small world second order network example with various clustering methods, and how to use these methods can be found in our TAC'17 paper Cheng/Kawano/S.'17.
- Applicable for second order equations description of the power grid.
- In another TAC'17 paper we develop a projection for exact model reduction for electrical networks with power loads based on a projected incidence matrix which provides a new decomposition of the Laplacian. Monshizadeh et al.'17.
- Relation with clustering as above, and network minimality not yet clear. Future work!

university of groningen

1 Introduction

2 Preliminaries

3 Clustering-based model reduction
- Structure-preserving projection
- Error bound
- Properties of the reduced model
- Some remarks
- Small world example

4 Clustering-based reduction of second order network systems

5 Generalized balancing

6 Conclusion and outlook

university of groningen

Background on balancing

- Balancing based on diagonalizing two Gramians. For asymptotically stable systems the controllability and observability Gramian.



university of groningen

Background on balancing

- Balancing based on diagonalizing two Gramians. For asymptotically stable systems the controllability and observability Gramian.
- Balanced truncation removes badly controllable and badly observable states, hence takes control system considerations into account.



university of groningen

Background on balancing

- Balancing based on diagonalizing two Gramians. For asymptotically stable systems the controllability and observability Gramian.
- Balanced truncation removes badly controllable and badly observable states, hence takes control system considerations into account.
- Balanced truncation preserves stability and balanced structure. However, other structure (such as network structure!) is generally not preserved.

university of groningen

Background on balancing

- Balancing based on diagonalizing two Gramians. For asymptotically stable systems the controllability and observability Gramian.
- Balanced truncation removes badly controllable and badly observable states, hence takes control system considerations into account.
- Balanced truncation preserves stability and balanced structure. However, other structure (such as network structure!) is generally not preserved.
- Network systems are semi-stable, thus balanced truncation not directly applicable. And how about the network structure?

  $\rightarrow$ Enforce network structure, and separate marginally stable part from asymptotically stable part.

university of groningen

Network structure preservation

- Reduce the system, possibly first destroying network structure.
- Make sure the reduced system can be transformed into a network system, i.e., when can a linear system be transformed into a network system through a state transformation?

$$\tilde{\Sigma}: \begin{cases} \dot{\hat{z}} = \mathcal{A}\hat{z} + \mathcal{B}u, \\ \hat{y} = \mathcal{C}\hat{z}, \end{cases} \qquad \hat{z} = T\hat{x} \iff \qquad \hat{\Sigma}: \begin{cases} \dot{\hat{x}} = -\hat{L}\hat{x} + \hat{F}u, \\ \hat{y} = \hat{H}\hat{x}, \end{cases}$$

university of groningen

Network structure preservation

- Reduce the system, possibly first destroying network structure.
- Make sure the reduced system can be transformed into a network system, i.e., when can a linear system be transformed into a network system through a state transformation?

$$\tilde{\Sigma}: \begin{cases} \dot{\hat{z}} = \mathcal{A}\hat{z} + \mathcal{B}u, \\ \hat{y} = \mathcal{C}\hat{z}, \end{cases} \qquad \hat{z} = T\hat{x} \iff \qquad \hat{\Sigma}: \begin{cases} \dot{\hat{x}} = -\hat{L}\hat{x} + \hat{F}u, \\ \hat{y} = \hat{H}\hat{x}, \end{cases}$$

Questions:

A. Which property of $\tilde{\Sigma}$ implies there exists a T s.t. $\hat{L}$ is a Laplacian?

B. How to ensure the system after balanced truncation has this property?

university of groningen

Network structure preservation

Focus on single integrator systems:

$$\tilde{\Sigma}: \begin{cases} \dot{\hat{z}} = \mathcal{A}\hat{z} + \mathcal{B}u, \\ \hat{y} = \mathcal{C}\hat{z}, \end{cases} \quad \hat{z} = T\hat{x} \iff \quad \hat{\Sigma}: \begin{cases} \dot{\hat{x}} = -\hat{L}\hat{x} + \hat{F}u, \\ \hat{y} = \hat{H}\hat{x}, \end{cases}$$

Theorem

$\tilde{\Sigma}$ is equivalent to a network system $\hat{\Sigma}$ (or $\mathcal{A}$ is similar to $-\hat{L}$)
if and only if all the eigenvalues of $\mathcal{A}$ are real, and one of its eigenvalues is zero, i.e.

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_{n-1}(A) < \lambda_n(A) = 0$$

university of groningen

Splitting up the system

First separate the zero eigenvalue of the Laplacian:

$$\boldsymbol{\Sigma}: \begin{cases} \dot{x} = -Lx + Fu, \\ y = Hx. \end{cases}$$

Semistable System

Transform the Laplacian:

$$L = T\Lambda_L T^T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_L & \\ & 0 \end{bmatrix} \begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix},$$

$$T_1 \in \mathbb{R}^{n \times (n-1)}, T_2 = \frac{\mathbf{1}}{\sqrt{n}}$$

university of groningen

Splitting up the system

First separate the zero eigenvalue of the Laplacian:

$$\Sigma : \begin{cases} \dot{x} = -Lx + Fu, \\ y = Hx. \end{cases}$$

Semistable System

$$z_s = T_1 x$$
$$z_a = T_2 x$$

$$\Sigma_s : \begin{cases} \dot{z}_s = Az_s + Bu, \\ y_s = Cz_s, \end{cases}$$
where $A = A^T = -\bar{\Lambda}_L$,

Stable System

Transform the Laplacian:

$$L = T\Lambda_L T^T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_L & \\ & 0 \end{bmatrix} \begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix},$$

$$T_1 \in \mathbb{R}^{n \times (n-1)}, \; T_2 = \frac{\mathbf{1}}{\sqrt{n}}$$

$$\Sigma_a : \begin{cases} \dot{z}_a = \frac{1}{\sqrt{n}} \mathbf{1}^T F u, \\ y_a = \frac{1}{\sqrt{n}} H \mathbf{1} z_a. \end{cases}$$

Average System

university of groningen

Generalized Gramians

Consider the stable subsystem:

$$\boldsymbol{\Sigma_s}: \begin{cases} \dot{z}_s = Az_s + Bu, \\ y_s = Cz_s, \end{cases} \quad \text{where } z_s \in \mathbb{R}^{n-1}$$

Generalized Controllability Gramian P and Observability Gramian Q

$$\min \operatorname{trace}(P)$$
$$AP + PA + BB^T \leq 0,$$

$$\min \operatorname{trace}(Q)$$
Q is diagonal;
$$A^T Q + QA + C^T C \leq 0.$$

university of groningen

Generalized Gramians

Consider the stable subsystem:

$$\Sigma_s : \begin{cases} \dot{z}_s = Az_s + Bu, \\ y_s = Cz_s, \end{cases} \quad \text{where } z_s \in \mathbb{R}^{n-1}$$

Generalized Controllability Gramian P and Observability Gramian Q

$$\min \operatorname{trace}(P)$$
$$AP + PA + BB^T \leq 0,$$

$$\min \operatorname{trace}(Q)$$
Q is diagonal;
$$A^T Q + QA + C^T C \leq 0.$$

Balancing Transformation such that in new coordinates:

$$TPT^T = T^{-T} Q T^{-1} = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{n-1}),$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n-1}$ are generalized Hankel singular values.

university of groningen

Generalized balanced truncation

- **Truncation:** If $\sigma_r \gg \sigma_{r+1}$, then truncation can be applied, i.e., put $x_{r+1} = \cdots = x_{n-1} = 0$.

university of groningen

Generalized balanced truncation

- **Truncation:** If $\sigma_r \gg \sigma_{r+1}$, then truncation can be applied, i.e., put $x_{r+1} = \cdots = x_{n-1} = 0$.
- Then combine the reduced system $\tilde{\Sigma}_s$ with the average system Σ_a (the part corresponding to the zero eigenvalue).

Result

All the eigenvalues of the reduced order system matrix $\hat{A}$ are real negative. Thus $\begin{bmatrix} \hat{A} & \\ & 0 \end{bmatrix}$ is similar to a Laplacian matrix, and thus there exists a transformation s.t. the reduced system is a network system.

university of groningen

Generalized balanced truncation

- **Truncation:** If $\sigma_r \gg \sigma_{r+1}$, then truncation can be applied, i.e., put $x_{r+1} = \cdots = x_{n-1} = 0$.
- Then combine the reduced system $\tilde{\Sigma}_s$ with the average system Σ_a (the part corresponding to the zero eigenvalue).

Result

All the eigenvalues of the reduced order system matrix $\hat{A}$ are real negative. Thus $\begin{bmatrix} \hat{A} & \\ & 0 \end{bmatrix}$ is similar to a Laplacian matrix, and thus there exists a transformation s.t. the reduced system is a network system.

The standard error bound now holds:

$$\|\Sigma - \hat{\Sigma}\|_\infty = \|\Sigma_s - \tilde{\Sigma}_s\|_\infty \leq 2 \sum_{i=r}^{n-1} \sigma_i,$$

university of groningen

Illustrative Example

For comparison, we consider an network example in Monshizadeh et al. '14 and Mlinarić et al.'15:

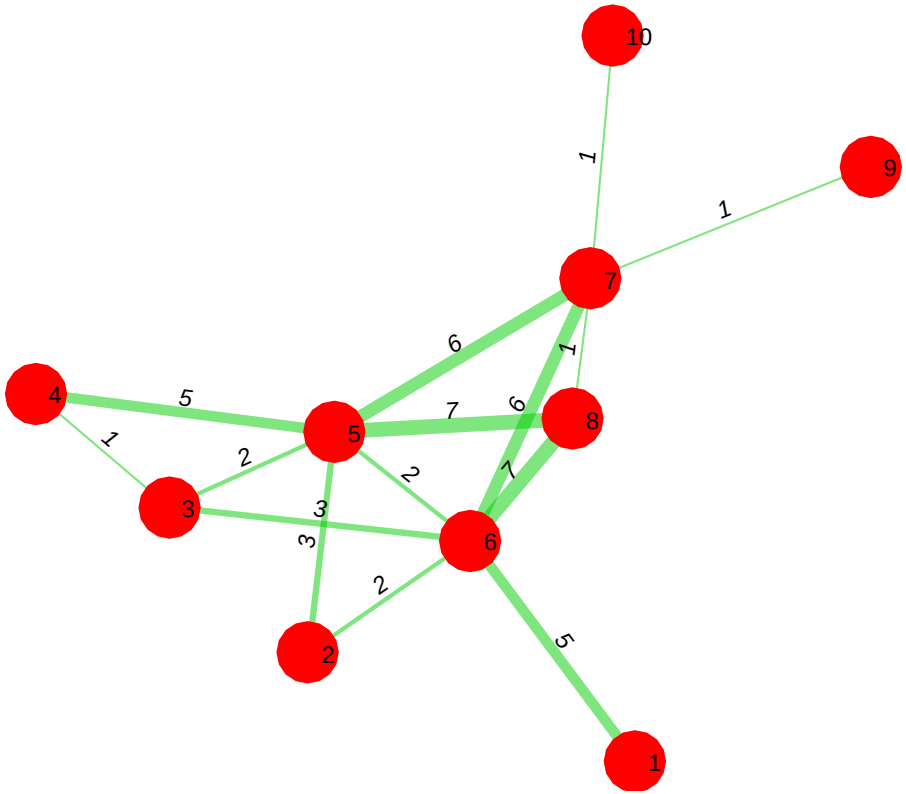


Figure: A network example with 10 vertices where the edge weights are shown by the thickness of the connection lines

university of groningen

Illustrative Example

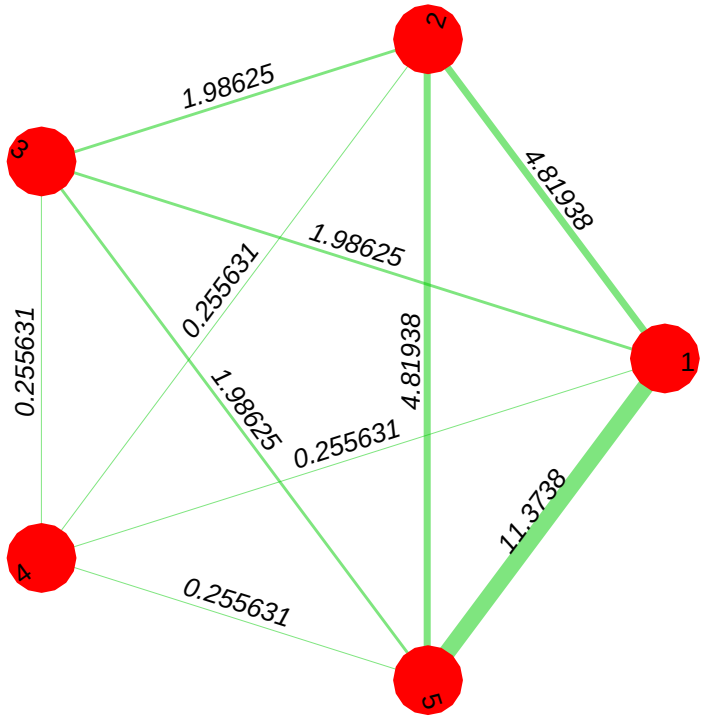
$$L = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 6 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 & 25 & -2 & -6 & -7 & 0 & 0 \\ -5 & -2 & -3 & 0 & -2 & 25 & -6 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & -6 & 15 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -7 & -7 & -1 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{L} = \begin{bmatrix} 18.4350 & -4.8194 & -1.9862 & -0.2556 & -11.3738 \\ -4.8194 & 11.8806 & -1.9862 & -0.2556 & -4.8194 \\ -1.9862 & -1.9862 & 6.2144 & -0.2556 & -1.9862 \\ -0.2556 & -0.2556 & -0.2556 & 1.0225 & -0.2556 \\ -11.3738 & -4.8194 & -1.9862 & -0.2556 & 18.4350 \end{bmatrix}$$



university of groningen

Illustrative Example



Approaches	Proposed Method	Monshizadeh	Mlinarić
$\frac{\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_2}}{\|\Sigma\|_{\mathcal{H}_2}}$	0.0849	0.5270	0.1459

university of groningen

Some remarks

- This balancing based model reduction procedure always allows to obtain a network system.



Some remarks

- This balancing based model reduction procedure always allows to obtain a network system.
- However, there is only a guarantee that the system can be represented by a complete graph, other structures cannot be guaranteed.

university of groningen

Some remarks

- This balancing based model reduction procedure always allows to obtain a network system.
- However, there is only a guarantee that the system can be represented by a complete graph, other structures cannot be guaranteed.
- Second order structure is not necessarily preserved either, potentially gradient system approach can help. S. /Van der Schaft'11

university of groningen

1 Introduction

2 Preliminaries

3 Clustering-based model reduction
- Structure-preserving projection
- Error bound
- Properties of the reduced model
- Some remarks
- Small world example

4 Clustering-based reduction of second order network systems

5 Generalized balancing

6 Conclusion and outlook

university of groningen

Conclusion

Summarizing:

- Structure preserving order reduction methods with error bounds based on clustering based on dissimilarity for first and second order networks.
- A generalized balancing approach that preserves the network structure is introduced.

university of groningen

Conclusion

Summarizing:

- Structure preserving order reduction methods with error bounds based on clustering based on dissimilarity for first and second order networks.
- A generalized balancing approach that preserves the network structure is introduced.

Various new concepts, such as network minimality, and a network Gramian are introduced. These are control systems concepts necessary for taking the reduction of networks a step further.

university of groningen

Outlook

- Directed graphs (one paper submitted).
- Nonlinear networks, Lur'e systems? Differential balancing? Kawano/S.'17

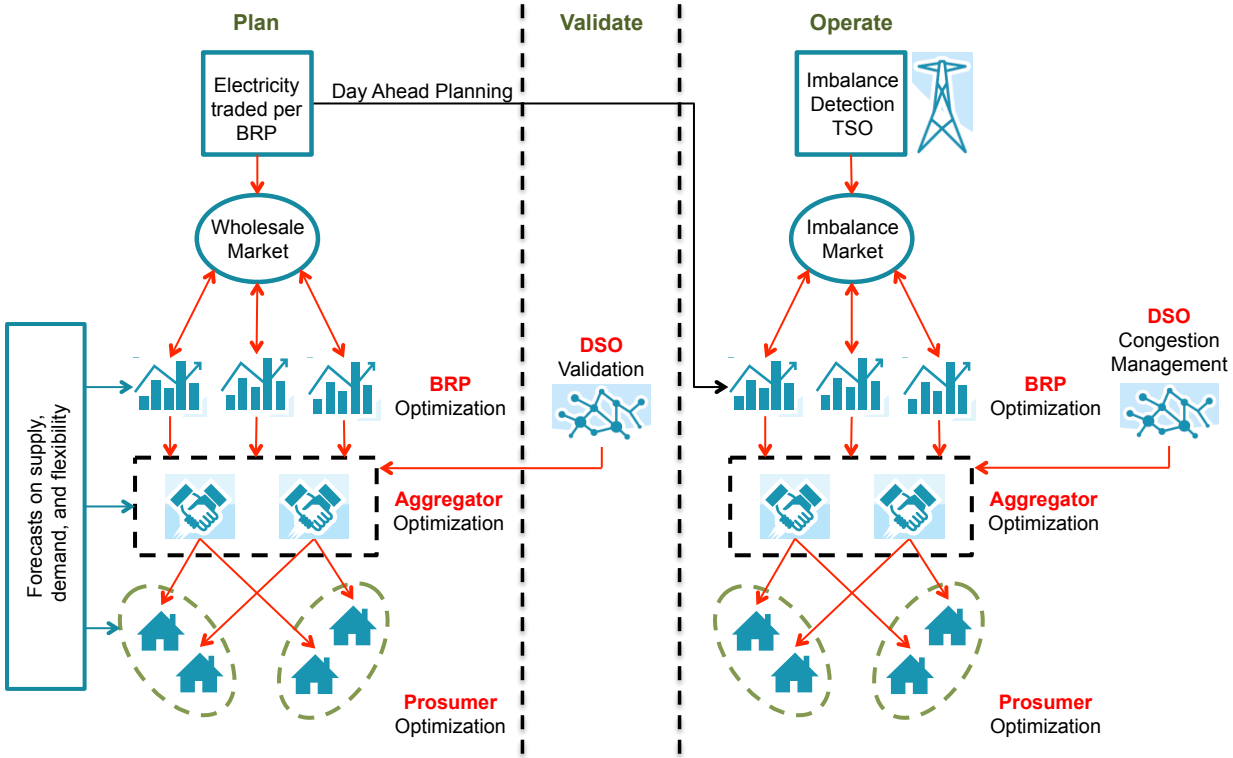
university of groningen

Outlook

- Directed graphs (one paper submitted).
- Nonlinear networks, Lur'e systems? Differential balancing? Kawano/S.'17
- Time scale separation in networks, very relevant for e.g., power networks. Slow coherency and singular perturbation methods should be incorporated.
- Other energy applications such as the different time scales of the power and gas networks and market structures? .

university of groningen

Outlook (power system market structure)



How to use this to reduce the complexity for hierarchical market networks for energy systems in combination with the various distributed optimization and optimal control approaches for such networks?

university of groningen

Bibliography



Xiaodong Cheng, Yu Kawano and Jacquelien M.A. Scherpen

Graph structure-preserving model reduction of linear network systems

ECC2016



Xiaodong Cheng, Yu Kawano and Jacquelien M.A. Scherpen

Clustering-based Model Reduction of Network Systems with Error Bounds

MTNS2016



Xiaodong Cheng, Jacquelien M.A. Scherpen and Yu Kawano

Structure-preserving Model Reduction of Second-order Network Systems

IEEE TAC 2017



Xiaodong Cheng, Jacquelien M.A. Scherpen

Controllability and Observability Gramians of Linear Network Systems

CDC2016



Xioadong Cheng, Jacquelien M.A. Scherpen

Balanced Truncation Approach to Linear Network System Model Order Reduction

to appear in Proc. IFAC 2017



Nima Monishizadeh, Claudio De Persis, Arjan van der Schaft, Jacquelien M.A. Scherpen

A Novel Reduced Model for Electrical Networks with Constant Power Loads

IEEE TAC 2017



Hildeberto Jardón Kojakhmetov, Jacquelien M.A. Scherpen

Model Order Reduction and Composite Control for a Class of Slow Fast Systems around a non- Hyperbolic Point

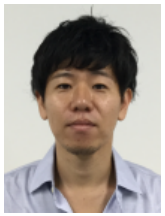
IEEE CSL 2017

For a full list: http://www.rug.nl/staff/j.m.a.scherpen

university of groningen

Acknowledgement







university of groningen