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Convergence Analysis and Control of Evolutionary Matrix-Game Dynamics

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Chapter 1

Introduction

1.1 Evolutionary game dynamics in populations

In social, economic, biological, technological, and various other types of networks, interconnected agents with simple dynamics may give rise to complex and seemingly unpredictable collective behaviors [163]. Traffic jams, market crashes, viral epidemics, and power blackouts are just a few examples of such collective behaviors having negative consequences. On the positive side, coordinated efforts such as volunteer disaster relief, free-market stabilization, and countless examples from biology provide a wealth of evidence that great challenges can be overcome through cooperation between individuals. Research questions in this topic naturally focus on how to predict and avoid the negative effects and how to promote the positive.

Another challenge in these types of networks is the promotion of cooperation among selfish individuals. Usually on one hand, there is a group task requiring the individuals to cooperate to optimize the collective performances, and on the other, each individual is self-interested and may prefer not to do so when she prioritizes her own interest, resulting in a social dilemma. It, therefore, becomes a fascinating yet formidable research question on how to reach a delicate balance between desired cooperation among agents as the team tasks require and logical self-interest of each agent ingrained to her instinct. As an illustration, in the famous *snowdrift game*, when a snowdrift blocks the road, each driver may choose to cooperate and clear the snow with the others or defect and burden the task on others' shoulders. There are also many engineering tasks such as environmental monitoring, smart manufacturing, and distributed information processing [29, 35, 38] that fit into such kind of social contexts. In the case of environmental monitoring for example, on one hand, we prefer to use as few sensors as possible to reduce energy consumption, and on the other hand, the environment must be monitored as the group task.

Unfortunately, localized analysis of the agent dynamics may reveal little about the underlying causes of these behaviors, in part because the major factors driving the collective dynamics may lie not in the individual agents but in the complex structure of their interconnections. Studying the system from a broader network perspective, perhaps subject to substantial simplification of the agent-level dynamics, can help to characterize critical properties such as controllability, convergence, stability, robustness, and performance [149]. Indeed we have seen a sweeping transition from local to network-based analysis across various disciplines in engineering and the physical and social sciences, which has led to many useful discoveries related to system dynamics on large com-

plex networks. For control scientists and engineers, these results present further opportunities to tackle large-scale social and economic issues in a control-theoretic framework.

One of the primary toolsets used to study these kinds of problems is *evolutionary game theory*. Originally proposed as a framework to study behaviors such as ritualized fighting in animals [160], evolutionary game theory has since been widely adopted in the social sciences. A key innovation of evolutionary game theory is that rather than assuming agents make perfectly rational choices, strategies and behaviors propagate through a population via some dynamic process that often does not require the agents to be rational. In the biological world, this propagation is manifested through *survival of the fittest* and reproductive processes, which are widely modeled using *population dynamics* [149] [181].

Under the assumptions that the population is infinite and well-mixed (completely connected), the population dynamics become a system of first-order differential equations, the most well-known of which are *replicator dynamics*. While these assumptions can lead to reasonable approximations for large, dense populations of organisms, there are many small societies, especially those involving humans, and many networks, in which the structure of the interactions plays a major role [117], leading to finite well-mixed and finite structured populations. In both of these cases, individuals revise their choices based on *update rules*, the two most prominent of which are *best-response* and *imitation*, resulting in the best-response and imitation dynamics [149, 155, 39]. Individuals interact with their neighbors by means of playing games, earn payoffs correspondingly, and revise their choices to the one that maximizes their payoffs against their neighbors if they follow the best-response update rule, and mimic their highest-earning neighbor if they follow the imitation update rule. It turns out that certain models of imitation reduce exactly to the replicator dynamics in the limit of the population size as it goes to infinity [153].

Researchers have found that network topology [141], phenotypic interactions [73, 138], repetition [172], punishment [175], population heterogeneity [136], emotion [139], as well as other components in game setups can all affect the success of cooperators in face of defectors. However, there remain many unanswered questions in each of these components. For example, although it is generally believed that networked interactions can help the maintenance of cooperation, neither the types of networks that lead to the maintenance, nor the asymptotic behavior of the dynamics under such networks are known. This is mainly due to lack of rigorous mathematical statements and dependence on simulation results. Additionally, in phenotypic interactions, the behavior of the phenotype-based decision-making individuals in different game setups is still an open topic. Moreover, although repeating the game played by the individuals in a population allows for conditional strategies and hence the promotion of cooperation by direct reciprocity, the level up to which cooperation is promoted remains unknown.

1.2 Problem statement

Given a population governed by some population dynamics, the key problem is to determine the asymptotic behavior of the dynamics, leading to convergence analysis. If one can establish such results, one of the next interesting problems is controlling the dynamics in order to reach a state where all or most of the individuals play a desired strategy. This can be done by, for example, offering payoff incentives to the agents, motivated by applications such as hastening the adoption

of new technologies or socially/environmentally beneficial behaviors and controlling the spread of viral epidemics [109].

The main problem studied in this thesis, followed by eight sub-problems, is stated in the following:

- to perform convergence analysis and provide control protocols for population dynamics in both infinite and finite, well-mixed and structured populations.

    1. How does a population of individuals playing repeated snowdrift games behave in the long run under the replicator dynamics? (Chapter 4)
    2. How does a population of individuals acting based on their phenotypes behave in the long run under the replicator dynamics? (Chapter 5)
    3. What is the asymptotic behavior of a finite well-mixed heterogeneous population governed by the best-response update rule? (Chapters 6 and 7)
    4. How to control a finite well-mixed heterogeneous population governed by the best-response update rule? (Chapter 6)
    5. What is the asymptotic behavior of a finite structured, heterogeneous population governed by the best-response update rule? (Chapter 8)
    6. How to control a finite structured, heterogeneous population governed by the best-response update rule? (Chapter 9)
    7. What is the asymptotic behavior of a finite structured, heterogeneous population governed by the imitation update rule? (Chapter 10)
    8. How to control a finite structured, heterogeneous population governed by the imitation update rule? (Chapter 11)

Investigating the first two sub-problems leads to the following mathematical sub-problems, all in the context of a continuously differentiable vector field:

1. What is the limit set of a point in a positively invariant simply-connected planar compact set that does not have any interior fixed point? (Chapter 2)
2. What role do the interior and boundary fixed points of a positively invariant simply-connected planar compact set play in forming the limit set of a point in the compact set? (Chapter 2)
3. What are the possible limit sets of a trajectory converging to a compact curve? (Chapter 3)
4. What are the possible limit sets of a trajectory converging to a compact manifold? (Chapter 3)

One of the mechanisms known to promote cooperation is *direct reciprocity* which is captured by *repeated games* where individuals play a base game repeatedly and can base their action in each round of the game on that of the opponent in the previous round, resulting in *reactive strategies*. While much research has investigated the performance of different reactive strategies under the prisoner's dilemma game [114, 69, 63, 56, 93, 68], less has been devoted to the snowdrift

game [134, 86, 178] despite the fact that snowdrift captures many behavioral patterns that cannot be well-modeled by the prisoner's dilemma [61]. Moreover, the strategies are usually compared pairwise [16, 17], and hence, the performance of different reactive strategies in a population where more than two are available remains an open problem. We address both issues in this thesis. While considering snowdrift as the base game, we focus on a population of individuals playing four reactive strategies, study the population under the replicator dynamics and let natural selection to choose the fittest strategies. We consider an arbitrary number of repetitions of the base game and present rigorous proofs for the convergence analysis of the resulting 3-dimensional dynamics.

Another mechanism known to be capable of promoting cooperation is having a cheap talk or preplay communication [46] before the game, during which players simultaneously send costless signals or messages to their opponents from a set available to each player before they play and consequently act based on the received messages [181]. This results in several decision rules, the most well-known of which are homophily, to cooperate only with similar others, and heterophily, to cooperate only with different others, both being widely studied in different setups [48, 138]. The evolution of the population shares of individuals following these and other decision rules has also been studied under death-birth population dynamics [80, 157, 59, 156, 60, 77], more or less claiming that preplay communication does not help to maintain cooperation in well-mixed populations and under the prisoner's dilemma game. The claim, however, remains without a mathematical proof and is explained only via examples and simulations. Moreover, the behavior of homophilics and heterophilics in exclusive populations of themselves or when only cooperators are involved remains an open problem under the prisoner's dilemma game. We aim to tackle both issues by revealing the innate properties of homophilic and heterophilic individuals under the replicator dynamics, and prove rigorously the simulation-based statement made in previous research.

Perception differences or heterogeneity of the individuals in a population is another factor influencing the cooperation level. Homogeneous populations, where each individual has the same tendency to cooperate, have been widely studied under best-response dynamics in different situations such as when the population is structured [187, 109, 110, 88, 92, 164, 61], the dynamics are noisy [84, 9, 8], and others [148, 123, 7, 99]. Although these works reveal interesting aspects of the best-response dynamics, or equivalently, linear threshold models, they ignore the impact of heterogeneity on the dynamics, which may cause complex features such as cooperation sustainability (the level-off phenomenon) [139]. To capture this factor, we relax the network, and consider a finite well-mixed heterogeneous population where each individual is associated with a possibly unique threshold or payoff matrix, chooses between two options: cooperation and defection or A and B, and updates her choice asynchronously according to the best-response update rule. We then investigate the asymptotic behavior of the dynamics, try to reveal the role of perception differences, and explain features such as the level-off phenomenon.

Perhaps the most influencing factor in the promotion of cooperation is the interaction network of the population. While the assumption of well-mixed populations makes analysis easier, the specific structure or topology of a network often plays a critical role in the dynamics. The primary goal in studying networked population dynamics is to determine the asymptotic behavior of the system, namely whether the dynamics converge to a single equilibrium state or limit cycle or fluctuate chaotically between several states. The convergence properties of networks with arbitrary structures has been investigated in the homogeneous (symmetric) case [110, 88, 92]. Moreover, in [37], a combination of mean-field approximations and simulations were used to show that syn-

chronous best-response dynamics in symmetric games tend to converge to an equilibrium state while imitation does not generally result in convergence to equilibria. In addition to making the convergence more likely, compared to synchronous models, asynchronous dynamics can provide a more realistic model of the time-line over which independent agents make decisions and receive information. Moreover, the heterogeneity of the population cannot be easily ignored, as explained above. So it remains to be seen under what conditions arbitrary networks of heterogeneous agents updating synchronously or asynchronously can be expected to converge to an equilibrium, and that is one of the goals of this thesis.

Finally, control of decision-making populations has become an attractive topic recently where researchers have started to try various methods to drive the population dynamics to a desired state. For example in [145, 144], the authors aim to find the minimum number of agents such that when these agents adopt a desired strategy under the imitation update rule, the rest of the agents in the network will follow. In the context of best-response and imitation dynamics, a natural mechanism for achieving strategy control is the use of payoff incentives. For instance, in [91], the payoffs of a stochastic snowdrift game are changed in order to shift the equilibrium to a more cooperative one. If the central agency can offer different rewards to each agent, a more efficient control protocol may be possible. That is, by altering the payoffs of just some individuals, the population can be led to a desired equilibrium state [144, 186]. As we will later discuss in Chapters 9 and 11, each of these methods leads to a particular control problem that desires its own solutions in the form of efficient algorithms, which we tackle in this thesis.

1.3 Approach and contribution

We handle the problems mentioned in the previous section by tools from dynamical systems, e.g., monotone properties of the vector field, game theory, e.g., Nash equilibrium, evolutionary game theory, e.g., convergence to Nash equilibrium and evolutionary stability, control theory, e.g., Lyapunov-like functions, and some that we develop by our own in this thesis, which inspired us to dedicate 3 chapters to these self-standing mathematical results: Chapter 2 where we tighten up the Poincaré-Bendixson Theorem and Chapter 3 where we show that trajectories converging to a curve in an arbitrary-dimensional continuously differentiable vector field, converge to the equilibrium points on that curve, and then extend the result to vector fields possessing an arbitrary-dimensional invariant compact manifold instead of a curve. In what follows, we provide more details on the conceptual approaches taken in this thesis.

To analyze the asymptotic behavior of the replicator dynamics under the repeated snowdrift game, we study two ratios of the four population shares and show that they exhibit a monotonic behavior; namely we can divide the state space into four areas, in each of which, each of the two ratios increases or decreases monotonically. This then implies that every interior trajectory of the system always converges to a line segment. On the other hand, as a separate result, we show in Chapter 3 that convergence to a simple open curve in general, implies convergence to the equilibrium points on that curve. Therefore, we know that interior trajectories will converge to an equilibrium on the line segment. Moreover, from evolutionary game theory, we know that if a trajectory converges to an equilibrium, it has to be a Nash equilibrium. Then by finding all Nash equilibria on the line segment, we determine the possible states to which an interior trajectory may

converge. Analysis of the boundary trajectories has been performed in previous studies, which we will briefly review.

We take a different approach for cheap-talk games under the replicator dynamics. We investigate the dominance relationships among the strategies, a notion taken from game theory, and use the fact that if a strategy is *weakly dominated* by a pure strategy, then the population share of either of the two must vanish in the long run, a result from evolutionary game theory. This enables us to determine which types of individuals survive in most of the population mixtures, yet it is not sufficient for investigating populations containing all four types, i.e., defectors, cooperators, homophilics and heterophilics. To carry out the analysis for such populations, we develop a general convergence theorem similar to the one on weakly dominated strategies, but this time the strategy may be weakly dominated only in the absence of some strategies whose associated population shares converge to zero. Using this result which can be applied more broadly to replicator dynamics with other type of games, we can easily determine the final survivors in the population.

In case of finite well-mixed heterogeneous populations, we mainly develop the necessary tools by our own, but the main idea of the proofs revolves around a discrete Lyapunov-like function that is lower bounded by zero and upper bounded by a positive constant. Moreover, the function's upper bound is tight and along the system's trajectory reaches and stays at its maximum after finite time. The function does not monotonically increase though. However, we construct an infinite set of time instants at each of which the function's lower bound becomes tightened up, enabling us to use this function as a Lyapunov-like function, and prove convergence of the dynamics to a particular set.

In case of finite networked heterogeneous populations, we again mainly develop the necessary tools by our own. For the best-response case, first by using a potential function, we prove that every homogeneous population of individuals with tendencies for choosing options A and B, will reach an equilibrium state in finite time. Then by using augmented graphs, we show how the result can be extended to a network of individuals with arbitrary tendencies. Namely, we add to the original network, nodes that balance the cooperation and defection tendencies of the individuals in the original network without modifying their dynamics. For the imitation case, we show that in populations where all individuals are *opponent coordinating*, i.e., earn a higher payoff if their opponents play the same strategy as they do, the highest payoff earned by the individuals does not decrease over time, and is upper bounded by some constant, resulting in a Lyapunov-like function. When this highest payoff settles at some value, we look at the second highest payoff in the population and show that it also converges to some fixed value, and by doing so show that the whole population eventually reaches some equilibrium state.

For control of finite networked heterogeneous populations, we start by developing a general framework for asynchronous network games with two available strategies, A and B. We define a network game to be *A-coordinating* if agents who update to strategy A would also do so if some agents currently playing B were instead playing A. Then we show that regardless of the update rule, providing incentives to the individuals when the network is at equilibrium leads the network to a unique equilibrium state, provided that the network game is A-coordinating. Both best-response and imitation update rules satisfy this condition for coordination games, and we use this fact to design efficient algorithms for the control problems in the corresponding context mentioned earlier.

The contribution of this thesis is three-fold: First, we show how different mechanisms including

repetition, phenotipic interaction and heterogeneity can help to maintain and promote cooperation under population dynamics. For the case of repeated games, the convergence analysis implies that the repetition of the base game results in final population states where individuals cooperate more often compared to the original coexistence of cooperators and defectors in populations playing the base game. While similar results have been claimed in populations containing fewer available reactive strategies or when the payoffs are non-parametric, here we verify our result for four reactive strategies and a fully parametric payoff matrix. This allows us to determine the range of parameters where the more-cooperative final population states show up, shedding light on how and under which condition repetition can actually promote cooperation. In the case of phenotipic interactions, we show homophilics have the tendency to battle over their phenotype, leading to a state where only one phenotype exists, while heterophilics show a more harmonic tendency, leading to a maximum diversity state. It then follows that cooperators can survive in face of homophilics, but will vanish against heterophilics. We also rigorously prove the incapability of preplay communication in maintaining cooperation in well-mixed populations containing defectors. In case of heterogeneous populations, we show that the state of the population dynamics always either fluctuates between two or reaches a single state where a number of cooperators and defectors coexist in the population. This coexistence in a well-mixed population is an impossible feature in homogeneous populations, which highlights the crucial role of heterogeneity in the maintenance of cooperation.

Second, by providing sufficient conditions for equilibrium convergence of networks governed by best-response and imitation update rules, we find factors that may cause non-converging behavior. Many real-life decision problems where one out of two actions must be chosen can be modeled on networks consisting of individuals who are either coordinating, that is, take an action only if sufficient neighbors are also doing so, or anticoordinating, that is, take an action only if too many neighbors are doing the opposite. It is not yet known whether such networks tend to reach a state where every individual is satisfied with his decision. We show that indeed any network of coordinating and any network of anticoordinating individuals always reaches a satisfactory state, regardless of how they are connected, how different their preferences are, and how many simultaneous decisions are made over time. These results reveal that irregular network topology, population heterogeneity, and partial synchrony are not sufficient to cause cycles or nonconvergence in populations governed by the best-response dynamics, or the linear threshold models, other factors such as imitation or the coexistence of coordinating and anticoordinating agents must play a role. By showing a similar result in populations governed by the imitation update rule but for a "smaller" class of networks, and showing that for a substantial class of other networks, the dynamics never converge, we conclude that convergence under imitation is in general a less likely phenomenon compared to the best-response dynamics.

Third, we provide control protocols for driving finite networked populations governed by the best-response and imitation update rules to desired equilibrium states. We show that networks governed by best-response dynamics are A-coordinating provided that each agent is coordinating and networks governed by imitation dynamics are A-coordinating provided that each agent is opponent-coordinating. We then proceed to four control problems some of which have been considered in previous research. The first is *uniform reward control* where a central agency has the power to uniformly change the payoffs of every agent by providing them a reward to increase their tendency to play a certain strategy, and the goal is to provide the minimum reward that

leads the network to a state where all play A. The second is *targeted reward control* where the regulating agency can target individual agents and offer them independent sufficient rewards to lead the network more efficiently to the desired state. The third is *budgeted targeted reward control* where the reward budget is limited and the goal is to maximize the number of A-playing individuals in the long run. The fourth is *direct strategy control* where the agency can directly control the strategies of the agents, and the goal is to find the minimum number of agents required to adopt the desired strategy, so that the whole network will eventually reach the desired state. Using the unique equilibrium convergence property that results from being A-coordinating, we design efficient algorithms for each of the above problems, and test their performance via several simulations.

1.4 Scope and outline of the thesis

As mentioned previously, in this thesis, we set out to both perform convergence analysis and provide control protocols for populations of decision-making individuals governed by either the best-response or imitation dynamics. We divide the analysis in three parts. We start with infinite well-mixed populations, leading to continuous-time dynamics in Part I, then proceed to finite well-mixed populations, leading to discrete-time dynamics in Part II, and finalize with finite structured populations, leading to again discrete-time, yet more complex dynamics in Part III. We elaborate on each part in detail in the following.

1.4.1 Part I: infinite well-mixed populations

In Part I, we focus on infinite, well-mixed populations, leading to continuous dynamics, the most well-known of which are the replicator dynamics. The results are not all restricted to the replicator dynamics though; namely, Chapters 2 and 3 are applicable to general continuously differentiable vector fields. The chapters appear more or less in an increasing order according to the dimension of the vector field investigated therein. We start with planar vector fields in Chapter 2 where we revisit the Poincaré-Bendixson theorem, then proceed to arbitrary dimensional vector fields possessing an invariant compact curve in Chapter 3 and find the possible limit sets of trajectories converging to the curve. Then we extend the result from a curve to an arbitrary-dimensional invariant compact set. Next, we proceed to the 4-dimensional replicator dynamics corresponding to a population of individuals playing repeated snowdrift games in Chapter 4. Finally, in Chapter 5, we investigate the arbitrary-dimensional replicator dynamics corresponding to a population of individuals having preplay communication in a prisoner's dilemma game.

1.4.2 Part II: finite well-mixed populations

In Part II, we focus on finite well-mixed populations, leading to discrete dynamics, the most well-known of which are the best-response dynamics, or equivalently linear threshold models. We consider a finite well-mixed heterogeneous population where each individual is associated with a possibly unique threshold or payoff matrix, chooses between two options A and B, or cooperation and defection, and updates her choice asynchronously according to the best-response update rule.

We start with when all agents' payoff matrices are that of either a prisoner's dilemma or a snowdrift game in Chapter 6, investigate possible control protocols of the model, and proceed to the case when all agent's payoff matrices are that of coordinating games in Chapter 7.

1.4.3 Part III: finite networked populations

In Part III, we focus on finite structured populations, leading to again discrete dynamics, two well-known of which are the best-response and the imitation dynamics. Correspondingly, the chapters are divided equally between these two dynamics: Chapters 8 and 9 investigate populations governed by the best-response dynamics and Chapters 10 and 11 investigate populations governed by the imitation dynamics. The chapters are also equally divided between the convergence analysis of the dynamics and the control of the dynamics: Chapters 8 and 10 are dedicated to the long run behavior of the dynamics while Chapters 9 and 11 are dedicated to the control of the dynamics.

1.5 List of publications

Conference papers:

1. Zhang, F., Ramazi, P. and Cao, M. "Distributed concurrent targeting for linear arrays of point sources." *In Proc. of the 19th IFAC World Congress*, pp. 8323-8328, Cape Town, South Africa, Aug. 2014.

2. Ramazi, P. and Cao, M. "Stability analysis for replicator dynamics of evolutionary snowdrift games." *In Proc. of the 53rd Conference on Decision and Control (CDC14)*, pp. 4515-4520, Los Angeles, USA, Dec. 2014.

3. Ramazi, P. and Cao, M. "Analysis and control of strategic interactions in finite heterogeneous populations under best-response update rule." *In Proc. of the 54th Conference on Decision and Control (CDC15)*, pp. 4537-4542, Osaka, Japan, Dec. 2015.

Journal papers:

1. Ramazi, P., Hessel, J. and Cao, M. "How feeling betrayed affects cooperation." *PloS One*, 10(4), p.e0122205, 2015.

2. Ramazi, P., Cao, M. and Weissing F.J. "Evolutionary dynamics of homophily and heterophily." *Scientific Reports*, 6:22766, 2016.

3. Ramazi, P., Riehl, J. and Cao, M. "Networks of conforming and non-conforming individuals tend to reach satisfactory decisions." *Proceedings of National Academy of Sciences (PNAS)*, 113(46), pp.12985-12990, 2016. (Chapter 8)

4. Ramazi, P., Jardon, H. and Cao, M. "Limit sets within curves where trajectories converge to." *Applied Mathematics Letters*, 68, pp.94-100, 2017. (Chapter 3)

5. Ramazi, P. Cao, M. and Scherpen, J. M. A. "Tightening Poincare-Bendixson theorem after counting separately the fixed points on the boundary and interior of a Planar region." *SIAM Journal on Applied Dynamical Systems*, under review. (Chapter 2)

6. Ramazi, P., Jardón-Kojakhmetov, H. and Cao, M. "Limit sets of trajectories converging to compact manifolds." *SIAM Journal on Applied Dynamical Systems*, under review. (Chapter 3)

7. Ramazi, P. and Cao, M. "Global convergence analysis for replicator dynamics of repeated snowdrift games." *SIAM Journal on Control and Optimization*, under review. (Chapter 4)

8. Ramazi, P. and Cao, M. "Asynchronous decision-making dynamics under best-response update rule in finite heterogeneous populations." *IEEE Transactions on Automatic Control*, under review. (Chapter 6)

9. Ramazi, P. and Cao, M. "Convergence of linear threshold decision-making dynamics in finite heterogeneous populations." *IEEE Transactions on Automatic Control*, under review. (Chapter 7)

10. Riehl, J., Ramazi, P. and Cao, M. "Efficient control of asynchronous best-response dynamics on networks through payoff incentives." *Automatica*, under review. (Chapter 9)

11. Ramazi, P. Riehl, J. and Cao, M. "Homophily, heterophily and the diversity of messages in cheap-talk games." To be submitted. (Chapter 5)

12. Ramazi, P., Riehl, J. and Cao, M. "Imitating successful neighbors hinders reaching satisfactory decisions." To be submitted. (Chapter 10)

13. Riehl, J., Ramazi, P. and Cao, M. "Control of asynchronous imitation dynamics on networks through payoff incentives." To be submitted. (Chapter 11)

1.6 Preliminaries

Consider a $\mathbf{C}^\ell$, $\ell \geq 1$, vector field

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n. \tag{1.1}$$

The flow generated by the vector field is denoted by $\varphi_t(\cdot) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and given a set $\mathcal{P} \subset \mathbb{R}^n$, $\varphi_t(\mathcal{P})$ is defined as $\bigcup_{p\in\mathcal{P}} \varphi_t(p)$. For a point $p \in \mathbb{R}^n$, the *orbit through* p, denoted by $\mathcal{O}(p)$, is defined as the set of points in the phase space that lie on the trajectory passing through p; more precisely, $\mathcal{O}(p) = \{x \in \mathbb{R}^n \,|\, x = \varphi_t(p), t \in \mathbb{R}\}$. Note that a fixed point x, i.e., $\phi_t(x) = x \ \forall t \in \mathbb{R}$, is also an orbit. By a *non-fixed orbit* we mean an orbit that is not a fixed point. Similarly a *non-fixed trajectory* is defined. The *positive semi-orbit of* p, denoted by $\mathcal{O}_+(p)$, is defined as the orbit of p induced by the flow $\varphi_t(p)$ for $t \geq 0$, i.e., $\mathcal{O}_+(p) = \{x \in \mathcal{P} \,|\, x = \varphi_t(p), t \geq 0\}$ [184], and $\mathcal{O}_-(p)$, *the negative semi-orbit*, is defined similarly but for $t \leq 0$.

The *boundary* of a set $\mathcal{S}$, denoted by $\mathrm{bd}(\mathcal{S})$, is the set of points p such that every neighborhood of p includes at least one point in $\mathcal{S}$ and one point out of $\mathcal{S}$, and the *interior* of $\mathcal{S}$, denoted by

$\text{int}(\mathcal{S})$, is the greatest open subset of $\mathcal{S}$. The closure of a set $\mathcal{S}$ is denoted by $\overline{\mathcal{S}}$. The distance from a point $p \in \mathbb{R}^n$ to a set $\mathcal{S} \subset \mathbb{R}^n$, denoted by $d(p, \mathcal{S})$, is defined by

$$d(p, \mathcal{S}) = \inf_{s \in \mathcal{S}} \|p - s\|$$

where $\|\cdot\|$ is taken as an arbitrary norm in $\mathbb{R}^n$. Correspondingly, the distance between two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined by $d(\mathcal{S}_1, \mathcal{S}_2) = \inf_{s_1 \in \mathcal{S}_1} d(s_1, \mathcal{S}_2)$. Given $x \in \mathbb{R}^n$, we say $\varphi_t(x)$ converges to a set $\mathcal{S} \subset \mathbb{R}^n$ as $t \to \infty$, and denote it by $\varphi_t(x) \to \mathcal{S}$ as $t \to \infty$, if and only if for any $\epsilon > 0$, there exists some $M > 0$ such that

$$d\left(\varphi_t(x), \mathcal{S}\right) < \epsilon \quad \forall t > M.$$

A set $\mathcal{S}$ is said to be invariant with respect to the vector field, if $s \in \mathcal{S}$ implies that $\mathcal{O}(s) \subseteq \mathcal{S}$, and positively invariant if $s \in \mathcal{S}$ implies that $\mathcal{O}_+(s) \subseteq \mathcal{S}$.

A point $q \in \mathbb{R}^n$ is called an *ω limit point of $p \in \mathbb{R}^n$*, if there exists a time sequence $\{t_i\} \to \infty$ such that $\{\varphi(t_i, p)\} \to q$. The set of all ω limit points of p is the *ω limit set of p*, denoted by $\omega(p)$. The α limit set, denoted by $\alpha(p)$, is defined similarly but by taking $\{t_i\} \to -\infty$ [57].

Lemma 1.1. *[76, refinement of Proposition 1.4.] Consider some point $p \in \mathbb{R}^n$ whose positive-semi orbit $\mathcal{O}_+(p)$ is bounded. Then $\omega(p)$ is nonempty, closed, connected and invariant under the vector field.*

1.7 Background on replicator dynamics

The replicator dynamics are described by [181, 149, 179]

$$\dot{x}_i = \left[u(e_i, x) - u(x, x)\right] x_i, \qquad i = 1, \ldots, n \tag{1.2}$$

where x, the vector obtained by stacking x_i together, belongs to the n-dimensional simplex, Δ, defined by

$$\Delta = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1,\ 0 \le x_i \le 1, i = 1, \ldots, n \right\},$$

e_i is the ith unit vector, also called the ith *vertex* of Δ, and $u : \Delta \times \Delta \to \mathbb{R}$ is the *utility function* defined by $u(x, y) = x^T A y$ with $A \in \mathbb{R}^{n \times n}$ being the *payoff matrix*. Since $u(\cdot, \cdot)$ is $\mathbf{C}^1$ in $\mathbb{R}^n$, it can be shown that (1.2) has a unique continuous solution [181, Proposition 3.20],[149, Section 4.A.2], we denote which by $x(t)$.

The following lemma shows that different payoff matrices may lead to the same replicator dynamics.

Lemma 1.2. *[181, pp 73] The replicator dynamics (1.2) are invariant under the addition of a constant to all the entries of any column of the payoff matrix A.*

Let $\mathcal{H}$ be a nonempty subset of $\{1, \ldots, n\}$. Then the convex hull of the unit vectors $e_i, i \in \mathcal{H}$, is called a *face* of Δ, and is denoted by $\Delta(\mathcal{H})$. The simplex Δ itself is a face; when $\mathcal{H}$ is proper, the face is called a *boundary face*, and when it has only two members, it is called an *edge* of Δ.

Lemma 1.3. *A face is invariant under the replicator dynamics.*

Proof. The proof uses some ideas from the proof of Proposition 3.20 in [181] where only the simplex is shown to be invariant. Let $\mathcal{H} \subseteq \{1, \ldots, n\}, \mathcal{H} \neq \emptyset$. First the invariance of the following set is shown:

$$\mathcal{Z} = \left\{ \sum_{i \in \mathcal{H}} \alpha_i e_i \,\middle|\, \alpha_i \geq 0 \, \forall i \in \mathcal{H} \right\}.$$

Let $x(0) \in \mathcal{Z}$. Then for every $j \in \{1, \ldots, n\}$,

$$x_j(0) = 0 \overset{(1.2)}{\Longrightarrow} x_j(t) = 0 \quad \forall t, \tag{1.3}$$

$$x_j(0) > 0 \overset{(1.3)}{\Longrightarrow} x_j(t) > 0 \quad \forall t, \tag{1.4}$$

where (1.4) can be shown by contradiction: assume on the contrary that $x_j(t_1) \leq 0$ for some $t_1 \in \mathbb{R}$. If $x_j(t_1) = 0$, then in view of (1.3), $x_j(0) = 0$, which is a contradiction. If on the other hand, $x_j(t_1) < 0$, then due to the continuity of $x(t)$, there exists some time t_2 such that $x_j(t_2) = 0$, which again leads to contradiction. Now (1.3) and (1.4) imply

$$x(t) \in \{z \in \mathbb{R}^n \,|\, z_i \geq 0 \ \forall i \in \mathcal{H}, \quad z_j = 0 \ \forall j \in \{1, \ldots, n\} - \mathcal{H}\} = \mathcal{Z} \qquad \forall t,$$

which proves the invariance of $\mathcal{Z}$.

Next, the invariance of the following set is shown:

$$\mathcal{Y} = \left\{ \sum_{i=1}^{n} \alpha_i e_i \,\middle|\, \sum_{i=1}^{n} \alpha_i = 1 \right\} = \{x \in \mathbb{R}^n \,|\, y(x) = 0\},$$

where $y(x) = \sum_{i=1}^{n} x_i - 1$. For this, it suffices to show that given $x \in \mathcal{Y}$, it holds that $\frac{\partial y(x)}{\partial x} \dot{x} = 0$, as indicated in the following:

$$\frac{\partial y(x)}{\partial x} \dot{x} = \sum_{i=1}^{n} \dot{x}_i = \sum_{i=1}^{n} u(e_i, x) x_i - \sum_{i=1}^{n} u(x, x) x_i = u(x, x) - u(x, x) \sum_{i=1}^{n} x_i = 0.$$

Finally, since both $\mathcal{Z}$ and $\mathcal{Y}$ are invariant, their intersection is also invariant and is given by

$$\mathcal{Z} \cap \mathcal{Y} = \left\{ \sum_{i \in \mathcal{H}} \alpha_i e_i \,\middle|\, \alpha_i \geq 0 \, \forall i \in \mathcal{H}, \sum_{i \in \mathcal{H}} \alpha_i = 1 \right\} = \text{convex-hull}\{e_i \,|\, i \in \mathcal{H}\} = \Delta(\mathcal{H}).$$

Hence, $\Delta(\mathcal{H})$ is invariant. □

The above result has several important implications: first, the simplex Δ is invariant; second, each vertex is a fixed point; third, each boundary face is invariant, implying that its dynamics can be determined independently from the rest of the simplex, and fourth, both the interior and the boundary of the simplex are invariant.

Part I

Infinite Well-Mixed Populations

Chapter 2

Tightening Poincaré-Bendixson Theorem after counting separately the fixed points on the boundary and interior of a planar region

This chapter tightens the classical Poincaré-Bendixson theorem for a positively invariant, simply-connected compact set $\mathcal{M}$ in a continuously differentiable planar vector field by further characterizing for any point $p \in \mathcal{M}$, the composition of the limit sets $\omega(p)$ and $\alpha(p)$ after counting separately the fixed points on $\mathcal{M}$'s boundary and interior. In particular, when $\mathcal{M}$ contains finitely many boundary but no interior fixed points, $\omega(p)$ contains only a single fixed point, and when $\mathcal{M}$ may have infinitely many boundary but no interior fixed points, $\omega(p)$ can in addition be a continuum of fixed points. When $\mathcal{M}$ contains only one interior and finitely many boundary fixed points, $\omega(p)$ or $\alpha(p)$ contains exclusively a fixed point, a closed orbit or the union of the interior fixed point and homoclinic orbits joining it to itself. When $\mathcal{M}$ contains in general a finite number of fixed points and neither $\omega(p)$ nor $\alpha(p)$ is a closed orbit or contains just a fixed point, at least one of $\omega(p)$ and $\alpha(p)$ excludes all boundary fixed points and consists only of a number of the interior fixed points and orbits connecting them. As an application of such tightening of Poincaré-Bendixson theorem, we carry out global convergence analysis for the planar case of a class of widely studied dynamical systems called replicator dynamics associated with evolutionary game theoretic models.

2.1 Introduction

Determining the asymptotic behavior of general continuous vector fields, even qualitatively, is still a daunting task. In the nineteenth century, Poincaré studied this problem for planar systems by focusing on the global behavior of the systems' trajectories without integrating the corresponding differential equations [128, 129, 130, 131, 36]. The related classical results are commonly referred to as the Poincaré-Bendixson Theorem [127, 21], which we summarize below. Consider the vector field

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^2 \tag{2.1}$$

where f is $\mathbf{C}^1$ on an open set $\mathcal{U}$ in $\mathbb{R}^2$. Denote the ω and α limit sets of a point p by $\omega(p)$ and $\alpha(p)$, respectively.

Theorem 2.1 (Standard form of Poincaré-Bendixson Theorem)**.** *[124, pp 245] For the vector field (2.1), suppose that there exists a point $p \in \mathcal{U}$, whose positive semi-orbit is contained in a compact subset of $\mathcal{U}$. Then if $\omega(p)$ contains no critical point of* (2.1)*, $\omega(p)$ is a periodic orbit of* (2.1)*.*

Similar other forms of the theorem can be found in [57, 64, 36], which are often used to establish the existence of periodic orbits, and thus not applied to the case when $\omega(p)$ contains fixed points. A more comprehensive version of the theorem reads as follows.

Theorem 2.2 (Comprehensive form of Poincaré-Bendixson Theorem)**.** *[184, Theorem 9.0.6] [76, Theorem 1.8] For the vector field (2.1), let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant complex for the vector field containing a finite number of fixed points. For any $p \in \mathcal{M}$, one of the following holds:*

1. *$\omega(p)$ is a fixed point;*
2. *$\omega(p)$ is a closed orbit;*
3. *$\omega(p)$ consists of a finite number of fixed points $p_1, \ldots, p_n$ and orbits γ with $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$, where p_i and p_j are not necessarily different. Moreover, for two distinct fixed points p_i and p_j, there exists at most one orbit γ such that $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$.*

From this theorem, although possibilities such as strange attractors and chaotic orbits can be easily ruled out, the third case in the theorem still gives rise to sometimes a large number of possible limiting behaviors. For example, when $\mathcal{M}$ contains just four fixed points on its boundary, there can be more than 300 different compositions of $\omega(p)$ even under the simplifying assumption that there is at most one honoclinic orbit at each fixed point. Some existing results have tried to reduce the possible scenarios; in [12, Theorem 68], the third case has been stated more precisely by stipulating that the trajectories γ must be the continuations of one another. Then for the example just mentioned, $\omega(p)$ can have more than 50 different compositions. In addition, one may tighten the theorem after knowing more properties of the vector field, e.g., being "relatively prime analytic". A planar vector field f is *relatively prime analytic* if the two components f_1 and f_2 of f (i) do not have a common analytic factor in any neighborhood of any point in $\mathbb{R}^2$ and (ii) have convergent power series in some neighborhood of every point in $\mathbb{R}^2$ [125]. By a *separatrix cycle* of (2.1) we mean a continuous image of a circle which consists of the union of a finite number of fixed points and compatibly oriented separatrices of (2.1), $p_j, \gamma_j, j = 1, \ldots, m$, such that for $j = 1, \ldots, m$, $\alpha(\gamma_j) = p_j$ and $\omega(\gamma_j) = p_{j+1}$ where $p_{m+1} = p_1$. A *graphic of* (2.1) is the union of a finite number of compatibly oriented separatrix cycles of (2.1). The following form of Poincaré-Bendixson theorem restricts the third case of Theorem 2.2 to a graphic.

Theorem 2.3 (Poincaré-Bendixson Theorem for a relatively prime analytic vector field)**.** *[124, pp 245, Theorem 3] Suppose that* (2.1) *is relatively prime analytic in $\mathcal{U}$. Consider $p \in \mathcal{U}$ and suppose that p's positive semi-orbit is contained in a compact subset of $\mathcal{U}$. Then one of the following holds:*

1. *$\omega(p)$ is a fixed point;*
2. *$\omega(p)$ is a closed orbit;*

3. $\omega(p)$ is a graphic of (2.1).

When the conditions of this theorem are satisfied, the example we considered just now of having $\mathcal{M}$ containing four fixed points on its boundary will guarantee that there is at most one honoclinic orbit at each fixed point and thus there are more than 50 possible outcomes of $\omega(p)$. This example shows that if one is interested in categorizing all possible asymptotic behaviors of a planar system qualitatively, one may still encounter difficulty even with the help of the existing most tightened form of Poincaré-Bendixson theorem. In the field of mathematical biology, when theoretical biologists try to predict the possible long-term evolutionary outcome of competing sub-populations using planar dynamical systems models, they run into many possible global phase portraits, each of which corresponds to a possible evolutionary outcome [24]. So there is great need in looking into how classical results, like Poincaré-Bendixson theorem, can be further tightened giving fewer and thus more tractable limiting behaviors under different assumptions about the specific properties of the system.

The aim of this chapter is to reduce the number of possible compositions of the limit sets of a vector field when knowing the number of fixed points on the boundary and in the interior of a given positively invariant, simply-connected compact set $\mathcal{M}$. We first investigate the case when $\mathcal{M}$'s interior contains no fixed point; we show that for any $p \in \mathcal{M}$, $\omega(p)$ must be a fixed point if $\mathcal{M}$ has a finite number of fixed points and must be either a fixed point or a continuum of fixed points if $\mathcal{M}$ has an infinite number of fixed points. In terms of the example given previously, such results imply that $\omega(p)$ in the example can only be one of the fixed points, so there are at most four possibilities. Then we proceed to the case when $\mathcal{M}$'s interior contains exactly one fixed point, and show that at least one of the ω or α limit sets is a fixed point, closed orbit or the union of the interior fixed points and homoclinic orbits[1] joining it to itself. Finally, we study the case of having a finite number of interior fixed points in $\mathcal{M}$. So the main contribution of the chapter is that the new results make it possible to provide more specific and tractable global convergence statements based on the counting of the fixed points on the boundary and in the interior of $\mathcal{M}$. To illustrate the effectiveness of our results, we apply our theorems to planar replicator dynamics that are popular models in theoretical biology to study evolutionary processes in large populations of interacting agents; we show that some known related results become much more straightforward to derive.

2.2 Main results

Before presenting our main results on tightened versions of Poincaré-Bendixson theorem, we first review some basic relevant results. The following lemma provides a sufficient condition for the existence of an ω limit set. It is applicable to higher dimensional spaces, but we restrict it here to the plane.

Lemma 2.1. *[184, Proposition 8.1.3] For the vector field* (2.1)*, let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set. Then for any point $p \in \mathcal{M}$, it holds that $\omega(p) \neq \emptyset$.*

[1] A homoclinic orbit is an orbit that starts from and ends at the same fixed point.

A continuous connected arc in the plane is said to be *transverse to the vector field*, if the vector field has no fixed points on the arc and nowhere becomes tangent to the arc [64]. By a *transverse* we refer to a closed line segment $\mathcal{L}$ that is transverse to the vector field. Due to the continuity of the vector field, clearly one can construct a transverse through any non-fixed point. The following lemma illustrates how the flow through a point p approaches a transverse through a non-fixed ω limit point $q \in \omega(p)$ when it exists.

Lemma 2.2. *[43, reformulation of Lemma 1.26] For the vector field* (2.1)*, consider a point $p \in \mathcal{U}$ such that $\mathcal{O}(p) \subset \mathcal{U}$. Let $q \in \omega(p)$ be a non-fixed point of* (2.1) *and let $\mathcal{L}$ be a transverse through q. Then there exists a sequence $\{t_i\} \to \infty$, such that $\{\phi(t_i, p)\} \in \mathcal{L}$ and $\{\phi(t_i, p)\} \to q$.*

The following result guarantees the existence of a fixed point inside a closed orbit, and is an immediate consequence of Index Theorem [184, Theorem 6.0.1].

Lemma 2.3. *[184, Corollary 6.0.2][57, Corollary 1.8.5] Enclosed by any closed orbit of* (2.1) *in $\mathcal{U}$, there must be at least one fixed point.*

Now we are ready to present the main results of the chapter.

2.2.1 $\mathcal{M}$ has no interior fixed point

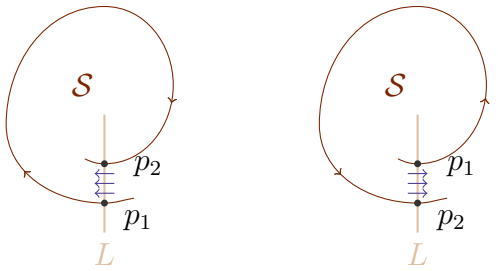
The following is the main result for the first case that we consider in this chapter.

Theorem 2.4 (No interior fixed points)**.** *For the vector field (2.1), consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points, all on* $\mathrm{bd}(\mathcal{M})$*. Then for any $p \in \mathcal{M}$, $\omega(p)$ is a fixed point on* $\mathrm{bd}(\mathcal{M})$*.*

Proof. From Theorem 2.2, it suffices to prove that $\omega(p)$ contains only fixed points since then only situation 1 is possible and the corresponding fixed point can only be on $\mathrm{bd}(\mathcal{M})$ as $\mathrm{int}(\mathcal{M})$ contains no fixed points. We prove this by contradiction, so assume on the contrary that there is a non-fixed point $q \in \omega(p)$. Then one can construct a transverse $\mathcal{L}$ through q, and from Lemma 2.2, we know that $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ for infinitely many times and such intersection points are in $\mathcal{M}$ since $\mathcal{O}_+(p) \subset \mathcal{M}$. So one can pick two consecutive intersection points p_1 and p_2 such that the line segment p_1p_2 lies in $\mathcal{M}$. Should p_1 and p_2 coincide, $\omega(p)$ would be a closed orbit, lying in $\mathcal{M}$, but encircling no fixed point as all the fixed points are on $\mathrm{bd}(\mathcal{M})$. This cannot happen in view of Lemma 2.3, and thus p_1 and p_2 must be distinct.

As illustrated by Figure 2.1, we construct the simply-connected compact set $\mathcal{S}$ whose boundary is formed by the semi-orbit $\mathcal{O}_+(p)$ from p_1 to p_2 and the line segment p_1p_2. Since $\mathcal{O}_+(p)$ always intersects $\mathcal{L}$ from the same side to the other, the orientation of the p_1-to-p_2 semi-orbit with respect to the line segment p_1p_2 must be one of the two cases shown in Figure 2.1. From the definition of $\mathcal{L}$, the vector field at any point on p_1p_2 intersects p_1p_2 from the same side of the line, and thus $\mathcal{S}$ is either positively invariant as shown in Figure 2.1.(a) or negatively invariant as shown in Figure 2.1.(b).

Since the boundary p_1-to-p_2 semi-orbit and p_1p_2 both lie in $\mathcal{M}$, we know that $\mathcal{S} \subseteq \mathcal{M}$. Hence, $\mathrm{int}(\mathcal{S}) \subseteq \mathrm{int}(\mathcal{M})$ and contains no fixed point. Moreover, neither $\mathcal{O}_+(p)$ nor $\mathcal{L}$ contains any fixed point, so $\mathrm{bd}(\mathcal{S})$ does not contain any fixed point. Therefore, $\mathcal{S}$ contains no fixed point. Consequently, if $\mathcal{S}$ is positively invariant, applying Theorem 2.2, we know that for any point $s \in \mathcal{S}$, $\omega(s)$



(a) (b)

Figure 2.1: The two possible cases for the positive semi-orbit $\mathcal{O}_+(p)$ in the proof of Theorem 2.4.

can only be a closed orbit confined in $\mathcal{S}$. But this contradicts Lemma 2.3. If on the other hand, $\mathcal{S}$ is negatively invariant, we apply the same argument after inverting the direction of the vector field and again reach the same contradiction. So the proof is complete. □

One might think that the condition in Theorem 2.4 requiring $\mathcal{M}$ to be simply connected is too strong and wonder what happens if $\mathcal{M}$ has holes in it. The following result explains that being simply connected follows naturally from the positive invariance property of $\mathcal{M}$ and the fact that its interior is empty of fixed points..

Proposition 2.1. *For the vector field (2.1), consider a connected compact set $\mathcal{M} \subset \mathcal{U}$ that is positively invariant. If the exterior boundary of $\mathcal{M}$ does not encircle any fixed points, then $\mathcal{M}$ is simply connected.*

We need the following result for the proof.

Lemma 2.4. *For the vector field* (2.1)*, if a set $\mathcal{S} \subset \mathcal{U}$ is negatively invariant, so is its closure $\overline{\mathcal{S}}$.*

Proof. We prove by contradiction. Assume on the contrary that $\overline{\mathcal{S}}$ is not negatively invariant. Then there exists $p \in \mathrm{bd}(\mathcal{S})$ such that $\phi(p, t_p)$ at some negative finite time $t_p < 0$ is bounded away from $\mathcal{S}$. We denote the distance from $\phi(p, t_p)$ to $\mathcal{S}$ by $d > 0$. It then follows from the continuity of the solutions with respect to the initial conditions (Theorem 3.5 in [78]) that there is a point $q \in \mathcal{S}$ that is close enough to p such that $\phi(q, t_p)$ is at least $\frac{d}{2}$ away from $\mathcal{S}$. But this contradicts the fact that $\mathcal{S}$ is negatively invariant. □

Proof of Proposition 2.1. We prove by contradiction. Assume on the contrary that $\mathcal{M}$ is not simply-connected. Let $\mathcal{J}$ be the smallest simply-connected compact set whose boundary $\mathrm{bd}(\mathcal{J})$ is $\mathcal{M}$'s exterior boundary. Then the set $\mathcal{J} - \mathcal{M}$ is nonempty and open since $\mathcal{M}$ is closed but not simply-connected. Furthermore, $\mathcal{J} - \mathcal{M}$ consists of one or more simply-connected, disjoint open sets. Let $\mathcal{A}$ be one of these sets, and then it must be negatively invariant since $\mathcal{M}$ is positively invariant. So in view of Lemma 2.4, $\overline{\mathcal{A}}$ is also negatively invariant. On the other hand, since $\mathcal{M}$'s exterior boundary does not encircle any fixed points, $\overline{\mathcal{A}}$ is free of any fixed point. Now if the vector field is reversed, $\overline{\mathcal{A}}$ becomes a positively invariant compact set free of any fixed point, which from Theorem 2.2 implies that for any $u \in \overline{\mathcal{A}}$, $\omega(u)$ is a closed orbit confined in $\overline{\mathcal{A}}$. But this contradicts Lemma 2.3. □

If in addition to being positively invariant, $\mathcal{M}$ is also negatively invariant, i.e., $\mathcal{M}$ is invariant, then Theorem 2.4 can get even more strengthened.

Theorem 2.5 (No interior fixed points)**.** *For the vector field (2.1), consider an invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points, all on* $\mathrm{bd}(\mathcal{M})$*. Then for any $p \in \mathcal{M}$, both $\omega(p)$ and $\alpha(p)$ are fixed points, not necessarily different, on* $\mathrm{bd}(\mathcal{M})$*.*

Proof. Theorem 2.4 implies that for any $p \in \mathcal{M}$, $\omega(p)$ contains only a single fixed point on $\mathrm{bd}(\mathcal{M})$. The same holds for $\alpha(p)$ after reversing the direction of the vector field since $\mathcal{M}$ is also negatively invariant. This completes the proof. □

2.2.2 $\mathcal{M}$ has no interior, but infinity many boundary fixed points

Theorem 2.2 requires that the vector field contains a finite number of fixed points, so it cannot be used in the situation in this subsection. Instead, we make use the following version of Poincaré-Bendixson Theorem that deals with the case when there are infinitely many fixed points. For a limit set $\omega(p)$, let $\omega_C(p)$ be the set of all fixed points in $\omega(p)$.

Theorem 2.6 (Poincaré-Bendixson Theorem for vector fields that may have infinitely many fixed points)**.** *[36, Theorem 6.1 with adjustment] For the vector field (2.1), let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set. Then for any $p \in \mathcal{M}$, one of the following two holds:*

1. *$\omega(p)$ is a closed orbit;*
2. *the set of those orbits that are not fixed points contained in $\omega(p)$ is at most countable; moreover, for any non-fixed point $q \in \omega(p)$, $\alpha(q)$ is contained in a connected subset of $\omega_C(p)$ and $\omega(q)$ is also contained in a connected subset of $\omega_C(p)$.*

Using Theorem 2.6, we obtain the following theorem that is the counterpart of Theorem 2.4 when the vector field may have infinitely many fixed points on $\mathrm{bd}(\mathcal{M})$:

Theorem 2.7. *For the vector field (2.1), consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that has no interior fixed point, but may contain an infinite number of fixed points on* $\mathrm{bd}(\mathcal{M})$*. Then for any $p \in \mathcal{M}$, one of the following two holds:*

1. *$\omega(p)$ is a fixed point on* $\mathrm{bd}(\mathcal{M})$*;*
2. *$\omega(p)$ is a continuum of fixed points on* $\mathrm{bd}(\mathcal{M})$*.*

Proof. Following similar steps as those in the proof for Theorem 2.4, one can construct the simply-connected compact set $\mathcal{S}$ as illustrated in Figure 2.1. Using similar arguments for $\mathcal{S}$ as those in the proof for Theorem 2.4, after applying Theorem 2.6, one knows that $\omega(p)$ does not contain any fixed point. On the other hand, $\omega(p)$ has to be connected [184, Proposition 8.1.3], so it can only be a connected subset of the fixed points in $\mathcal{M}$, which is either a fixed point or a continuum of fixed points on $\mathrm{bd}(\mathcal{M})$. □

In the following subsection, we look into the situation when $\mathcal{M}$ has one and only one interior fixed point.

2.2.3 $\mathcal{M}$ has exactly one interior fixed point

Now we present the counterpart of Theorem 2.4 discussing the case when $\mathcal{M}$ contains exactly one interior and finitely many boundary fixed points.

Theorem 2.8 (one interior fixed point)**.** *For the vector field (2.1), consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains exactly one interior fixed point x^* and a finite number of fixed points on its boundary. Then for any $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit encircling x^* or the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself;*
2. *$\alpha(p)$ is $\{x^*\}$, a closed orbit encircling x^* or the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself.*

Proof. We investigate all possibilities for $\omega(p)$ and show that each results in one of the cases of the theorem. Should $\omega(p)$ be a singleton fixed point or a closed orbit that has to encircle g according to Lemma 2.3, we arrive at Part 1. of the theorem. So consider the situation when $\omega(p)$ is neither. It then follows Theorem 2.2 that $\omega(p)$ contains non-fixed points; we pick one such point q and construct a transverse $\mathcal{L}$ through q. From Lemma 2.2, we know that $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ for infinitely many times. Consider two consecutive intersections p_1 and p_2 which have to be distinctive since $\omega(p)$ is not a closed orbit. We construct the simply-connected compact set $\mathcal{S}$ whose boundary is formed by the semi-orbit $\mathcal{O}_+(p)$ from p_1 to p_2 and the line segment p_1p_2. Similar to the proof of Theorem 2.4, one can show that:

(i) $\mathcal{S}$ is in the form of one of the two cases shown in Figure 2.1,

(ii) $\mathcal{S}$ is positively invariant in Case (a) of the figure and negatively invariant in Case (b), and

(iii) $x^* \in \text{int}(\mathcal{S})$ is the only fixed point in $\mathcal{S}$.

If $\mathcal{S}$ is positively invariant, $\mathcal{O}_+(p) \cap \text{int}(\mathcal{S}) \neq \emptyset$, implying the existence of some $s_p \in \mathcal{O}_+(p) \cap \text{int}(\mathcal{S})$. Consequently, $\omega(s_p) = \omega(p)$. Then, applying Theorem 2.2, we know that $\omega(s_p)$ consists of a number of fixed points in $\mathcal{S}$ and the orbits connecting them. However, since x^* is the only fixed point in $\text{int}(\mathcal{M})$, such orbits can only connect x^* to itself. So $\omega(s_p)$ is the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself, so is $\omega(p)$. So in this case Part 1 of the theorem holds.

Otherwise, if $\mathcal{S}$ is negatively invariant, then there exists a point $s_p \in \mathcal{O}_-(p) \cap \text{int}(\mathcal{S})$ where $\mathcal{O}_-(p)$ is the same as $\mathcal{O}_+(p)$, but when time is reversed. Consequently, after reversing the direction of the vector field, one can check the three cases in Theorem 2.2 as $\omega(s_p)$ lead to the three cases in Part 2 of the theorem respectively. □

Theorem 2.8 is indeed restricting the third case of Theorem 2.2, for at least one of the ω or α limit sets. Note that if in addition x^* is hyperbolic[2] and the vector field contains no closed orbits, then for any point $p \in \mathcal{M}$, either $\omega(p)$ is a fixed point or $\alpha(p) = \{x^*\}$. It is also worth mentioning that some cases in Part 1 and Part 2 of Theorem 2.8 never take place at the same time. For example, it is impossible to have both $\omega(p)$ and $\alpha(p)$ being the union of $\{x^*\}$ and homoclinic orbits joining x^* to itself. We exclude such cases for general positively invariant compact regions as follows. We call a point on a closed orbit a *periodic* point.

[2]A fixed-point x^* is said to be hyperbolic if every eigenvalue of the Jacobian matrix of the vector field at x^* has a nonzero real part.

Proposition 2.2. *Let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set under the vector field (2.1). For any non-periodic point $p \in \mathcal{M}$, if $\omega(p) = \alpha(p)$, then the limit sets contain only fixed points.*

For the proof, we need the following lemma.

Lemma 2.5 (Lemma 9.0.2 [184])**.** *Let $\mathcal{L} \subset \mathcal{M}$ be a transverse to the vector field. Then for any point $p \in \mathcal{M}$, $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ in a monotone sequence; that is, if p_i, $i > 1$, is the ith intersection of $\mathcal{O}_+(p)$ with $\mathcal{L}$, then $p_i \in [p_{i-1}, p_{i+1}]$.*

Proof of Proposition 2.2. We prove by contradiction. Assume on the contrary that $\omega(p)$ includes a non-fixed point q. Then we can construct a transverse $\mathcal{L}$ through q, which in view of Lemma 2.2, has infinitely many intersection points with $\mathcal{O}_+(p)$. Consider two consecutive intersections p_i and p_{i+1} which are distinctive since p is not on a closed orbit. We construct the simply-connected compact set $\mathcal{S}$ whose boundary is formed by the semi-orbit $\mathcal{O}_+(p)$ from p_1 to p_2 and the line segment p_1p_2, resulting in either Case (a) or (b) in Figure 2.1. First we look at Case (a). From Lemma 2.5, we know $q \in \text{int}(\mathcal{S})$ since otherwise, the intersections of $\mathcal{O}_+(p)$ with $\mathcal{L}$ do not converge to q. On the other hand, $\mathcal{S}$ is positively invariant, implying that $\alpha(p) \cap \mathcal{S} = \emptyset$. Hence, $q \notin \alpha(p)$. Now we look at Case (b). In view of Lemma 2.5, $q \in \mathcal{M} - \mathcal{S}$. On the other hand, $\mathcal{S}$ is negatively invariant, implying that $\alpha(p) \subseteq \mathcal{S}$. Hence, again $q \notin \alpha(p)$. However, this contradicts the assumption $\omega(p) = \alpha(p)$, which completes the proof. □

In case $\mathcal{M}$ contains finitely many fixed points, we can sharpen the result of Proposition 2.2.

Corollary 2.1. *For the vector field (2.1), let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set containing a finite number of fixed points. Then for any non-periodic point $p \in \mathcal{M}$, if $\omega(p) = \alpha(p)$, then the limit sets exclusively contain a single fixed point.*

Proof. In view of Proposition 2.2, $\omega(p)$ and $\alpha(p)$ contain only fixed points. On the other hand, $\omega(p)$ is connected [184, Proposition 8.1.3]. Hence, $\omega(p)$ and $\alpha(p)$ are either a fixed point or a continuum of fixed points. The latter is impossible since there are only a finite number of fixed points in $\mathcal{M}$. □

In the following section, we look at the situation when there are a finite number of fixed points in $\text{int}(\mathcal{M})$.

2.2.4 $\mathcal{M}$ has finitely many interior fixed points

Following the previous subsection of having one interior fixed point in the positively invariant compact set $\mathcal{M}$, we now extend the result to the more general case of having finitely many interior fixed points in $\mathcal{M}$.

Theorem 2.9 (finitely many interior fixed points)**.** *For the vector field (2.1), consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ containing a finite number of fixed points. Then for any point $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit encircling at least one interior fixed point or the union of some interior fixed points together with the orbits connecting them;*

2. *$\alpha(p)$ is an interior fixed point, a closed orbit encircling at least one interior fixed point or the union of some interior fixed points together with the orbits connecting them.*

Proof. The proof is similar to that for Theorem 2.8 and we omit it here. □

Compared to the classical form of Poincaré-Bendixson Theorem 2.2, what Theorem 2.9 has further clarified is the role of the interior fixed points of $\mathcal{M}$ play to influence the topological structure of the limit sets. For example, as an immediate result of Theorem 2.9, if the third case of Theorem 2.2 takes place for p, then $\omega(p)$ and $\alpha(p)$ cannot be free of interior fixed points at the same time; in other words, unless $\omega(p)$ is simply a fixed point or a closed orbit, some interior fixed points must be in either $\omega(p)$ or $\alpha(p)$. Another implication of Theorem 2.9 is the exclusion of the boundary fixed points from one of $\omega(p)$ and $\alpha(p)$. From Theorem 2.9, if $\omega(p)$ is not simply a fixed point, then at least one of $\omega(p)$ or $\alpha(p)$ does not contain any boundary fixed point. In a sense, this implies that the interior fixed points are more important for determining the structures of the limit sets. Finally, we note that Corollary 2.1 can also be utilized here to rule out some trivial cases when $\omega(p)$ and $\alpha(p)$ are the same.

At the end of this section, we present the following version of Theorem 2.9 without requiring $\mathcal{M}$ to be simply connected.

Theorem 2.10. *For the vector field* (2.1)*, consider a positively invariant, compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points. Then for any $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit or the union of some interior fixed points with the orbits connecting them;*

2. *$\alpha(p)$ is one of the interior fixed points, a closed orbit or the union of some interior fixed points with the orbits connecting them.*

Proof. The proof is similar to that of Theorem 2.8. The difference is that if $\omega(p)$ or $\alpha(p)$ is a closed orbit, it may encircle areas that do not belong to $\mathcal{M}$. □

In the following section, we discuss how to apply the tightened results for Poincaré-Bendixson theorem to replicator dynamics, an important class of population dynamics as models for theoretical biologists, economists and sociologists.

2.3 Application to replicator dynamics

Because of the constraint $x \in \Delta$, the n-dimensional replicator dynamics (1.2) give rise to in fact an $(n-1)$-dimensional vector field. In this section, we consider the planar replicator dynamics, that is when $n = 3$. Except for a few trivial cases, the planar replicator dynamics usually have at most one interior fixed point in the planar simplex, which reduces to a triangular area, making it a suitable case study for the applications of the results developed in the previous section. The

corresponding payoff matrix is a 3-by-3 real matrix with 9 parameters. However, in view of Lemma 1.2, one can transform the payoff matrix into the following form with 6 parameters

$$A = \begin{bmatrix} 0 & -b & -f \\ a & 0 & c-f \\ d & e-b & 0 \end{bmatrix}, \qquad a, b, c, d, e, f \in \mathbb{R},$$

without changing the replicator dynamics. We are interested in the asymptotic behavior of (1.2), particularly the limit sets of an arbitrary point in Δ. As discussed above, each edge is invariant, and thus exhibits a one-dimensional flow that can be easily determined. Therefore, we focus on the limit sets of Δ's interior points. In [24] (later corrected in [25]), the possible phase portraits of such planar dynamics are classified into 49 qualitatively different cases. The classification is made according to the locations of the fixed points, i.e., (i) more than two fixed points in int(Δ), not all on a straight line, (ii) more than one fixed point in int(Δ), all on one straight line, (iii) exactly one fixed point in int(Δ) and (iv) no fixed point in int(Δ). The fourth class leads to most of the possible phase portraits, and is itself classified into the following sub-cases: (iv-a) two edges point-wise fixed, (iv-b) one edge point-wise fixed and (iv-c) no edge point-wise fixed. For the last sub-case (iv-c), i.e., when there is no fixed point in int(Δ) and no edge is point-wise fixed, the author of [24] made the following statement in Section III.c: "*... we can deduce that on every edge there is at least one corner which has a non-vanishing EV(eigenvalue) in direction of that edge, whence we can infer that only robust flows on the edges occur. (new paragraph) As a consequence of Poincaré-Bendixson theory, the only possible robust PP (phase portrait)'s are 34 to 43.*" Here PP 34 to 43 are referring to those in Figure 2.5 of [24].

However, as discussed in the introduction, for each of the phase portraits just referred to, Poincaré-Bendixson theorem itself usually cannot give the description of the global convergence only based on some local stability analysis, which prevents to reach directly the quoted conclusions in [24]. Instead, one has to rely on results like those developed in the previous section to draw such conclusions. We illustrate this point by deriving the phase portrait of (1.2) for three example payoff matrices as follows.

Example 2.1. Consider the following payoff matrix

$$A_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ -3 & 0 & 3 \end{bmatrix}. \tag{2.2}$$

Then for (1.2), as shown in Figure 2.2(a), the fixed points are e_1 and q_2 which are hyperbolic saddle points, e_3 which is hyperbolic stable and e_2 which is a center fixed point. The flow on the edges is also shown in Figure 2.2(a). Consider a point p on the intersection of int(Δ) and the stable invariant manifold of q_2. Then $\alpha(p)$ has to be a fixed point from Corollary 2.5. However, the only point in the simplex that a trajectory may converge to as $t \to -\infty$ is e_2. Hence, $\alpha(p) = e_2$. So there exists an orbit γ that starts from e_2 and ends at q_2 (Figure 2.2(b); indeed γ can be shown to be a straight line). Now the area $\mathcal{Z}_1$ surrounded by γ, the line segment q_2e_3 and the edge e_3e_2 is an invariant simply-connected compact set that does not include any interior fixed point. Hence, from Corollary 2.5, every orbit in $\mathcal{Z}_1$ starts from and ends at a fixed point. On the other hand, q_2 is a hyperbolic saddle, and thus the only orbit that converges to it is γ. Hence, every orbit in

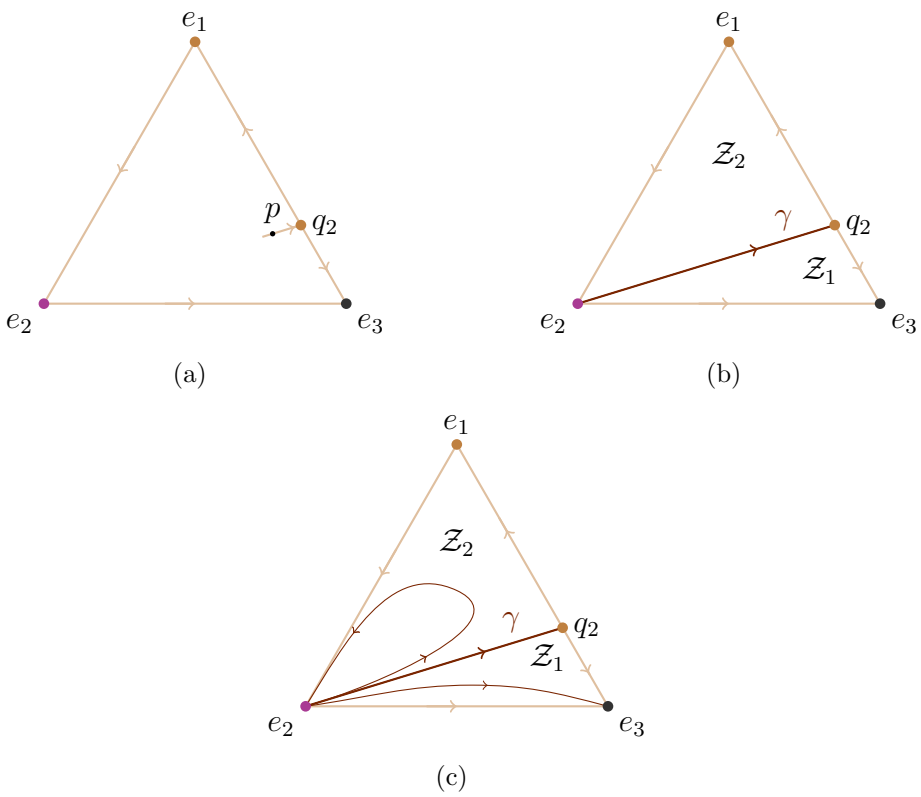


Figure 2.2: Deriving the phase portrait of the replicator dynamics with the payoff matrix A_1 in (2.2). (a) e_1 and q_2 are hyperbolic saddle, e_3 is a hyperbolic stable and e_2 is a center fixed point. p is a point on the stable invariant manifold of q_2. (b) In view of Corollary 2.5 and because of the local stability results, $\alpha(p) = e_2$. This results in the trajectory γ that divides Δ into $\mathcal{Z}_1$ and $\mathcal{Z}_2$. (c) Again, Corollary 2.5 and the local stability results imply that for each $z \in \text{int}(\mathcal{Z}_1)$, $\alpha(z) = e_2$ and $\omega(z) = e_3$, and for each $z \in \text{int}(\mathcal{Z}_2)$, $\alpha(z) = \omega(z) = e_2$.

$\text{int}(\mathcal{Z}_1)$ starts from e_2 and ends at e_3, namely for all $z \in \text{int}(\mathcal{Z}_1)$, $\alpha(z) = e_2$ and $\omega(z) = e_3$ (Figure 2.2(c)). Now consider the other part of the simplex, i.e., the area $\mathcal{Z}_2$ surrounded by γ, the line segment q_2e_1 and the edge e_1e_2. Again, in view of Corollary 2.5, for all $z \in \text{int}(\mathcal{Z}_2)$, $\omega(z)$ and $\alpha(z)$ are fixed points. However, neither e_1 nor q_2 can be the limit set of any point $z \in \text{int}(\mathcal{Z}_2)$. So for all $z \in \text{int}(\mathcal{Z}_2)$, $\omega(z) = \alpha(z) = \{e_2\}$, namely, every orbit in $\text{int}(\mathcal{Z}_2)$ is a homoclinic orbit joining e_2 to itself (the resulting phase portrait matches PP 45 in [24]). □

Example 2.2. Consider the following payoff matrix

$$A_2 = \begin{bmatrix} 0 & 3 & -1 \\ -1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}. \tag{2.3}$$

Then for (1.2), as shown in Figure 2.3(a), the fixed points are e_1, e_3 and q_1 which are hyperbolic saddle points, e_2 which is hyperbolic unstable, and g which is hyperbolic stable. The flow on the edges is also shown in Figure 2.3(a). From [65] we know that no limit cycle shows up in

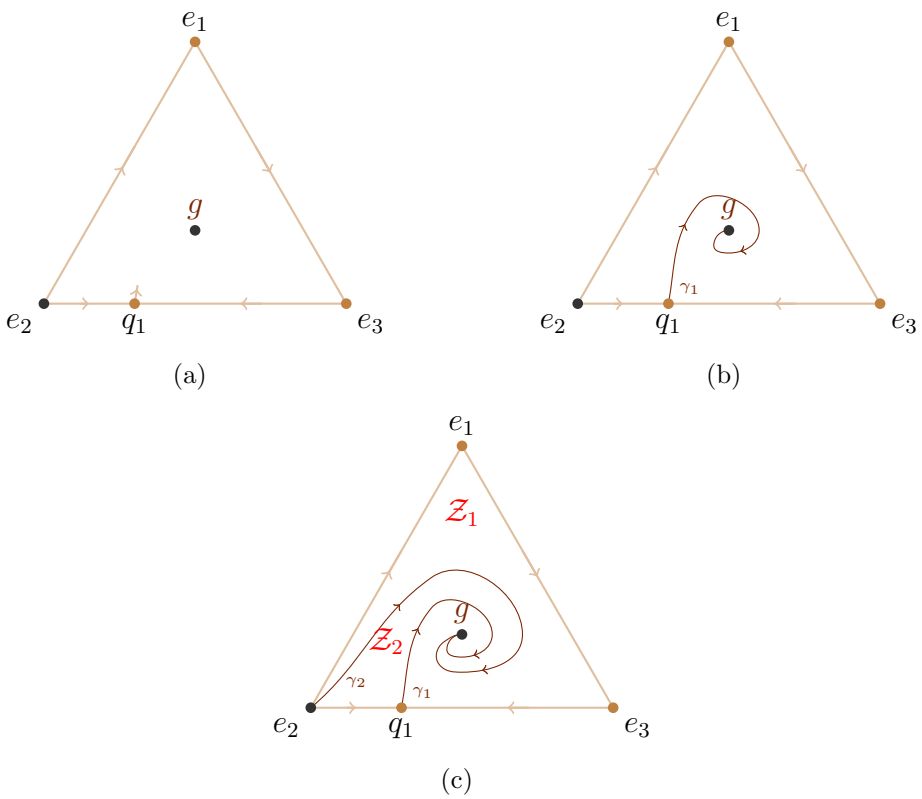


Figure 2.3: Deriving the phase portrait of the replicator dynamics with the payoff matrix A_2 in (2.3). (a) e_1, e_3 and q_1 are hyperbolic saddle, e_2 is a hyperbolic unstable, and g is a hyperbolic stable fixed point. (b) Because of Corollary 2.8, the local stability results and the fact that no limit cycle exists, $\omega(p) = \{g\}$ for all $p \in \text{int}(\Delta)$. Hence, the unique out-going trajectory from q_1, denoted by γ_1, converges to g. (c) The rest of the orbits in $\text{int}(\Delta)$ start from e_2 and end at g. This is because any out-going trajectory from e_2, e.g., γ_2, together with γ_1 divide the simplex into the zones $\mathcal{Z}_1$ and $\mathcal{Z}_2$, each of which satisfy the condition of $\mathcal{M}$ in Corollary 2.8. Hence, every trajectory in $\text{int}(\mathcal{Z}_i), i = 1, 2$, start from e_2 and end at g.

3-dimensional replicator dynamics. Hence, only trivial closed orbits around g may be observed. However, this is not possible since g is stable. Hence, no closed orbit exists. Further, since g is stable, no homoclinic orbit exists that joins g to itself. Hence, in view of Theorem 2.8, for any point $p \in \text{int}(\Delta)$, either $\omega(p)$ is a fixed point, or $\alpha(p)$ is $\{g\}$. The latter, however, is not possible since g is stable. Hence, $\omega(p)$ is a fixed point. On the other hand, the only fixed point, to which convergence is possible, is g. Hence,

$$\forall p \in \text{int}(\Delta), \quad \omega(p) = \{g\}. \tag{2.4}$$

So the unique orbit γ_1 that lies on the intersection of $\text{int}(\Delta)$ and the unstable invariant manifold of q_1 converges to g, (Figure 2.3(b)). Consider an arbitrary out-going orbit from the unstable fixed point e_2, and denote it by γ_2. In view of (2.4), γ_2 converges to g. As shown in Figure 2.3(c), γ_1 and γ_2 divide the simplex into areas $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Each area is an invariant simply-connected compact set. Hence, from Corollary 2.5, for any point $z \in \text{int}(\mathcal{Z}_i), i = 1, 2$, $\alpha(z)$ is a fixed point. On the

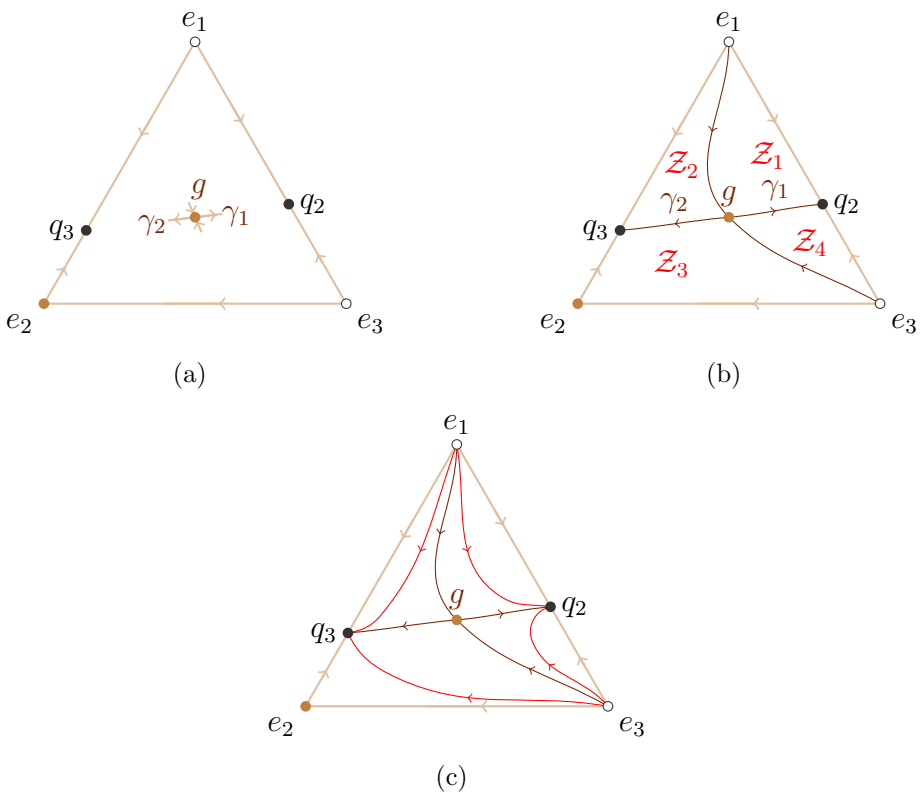


Figure 2.4: Deriving the phase portrait of the replicator dynamics with the payoff matrix A_3 in (2.5). (a) e_2 and g are hyperbolic saddle, e_1 and e_3 are hyperbolic unstable and q_3 and q_2 are hyperbolic stable fixed points. The trajectories γ_1 and γ_2 lie on the unstable invariant manifold of g. (b) Because of Corollary 2.8 and the local stability results, the unstable invariant manifold of g is confined to q_2 and q_3 and the stable invariant manifold of g is confined to e_1 and e_3. This results in the four zones $\mathcal{Z}_1, \dots, \mathcal{Z}_4$. (c) In view of Theorem 2.4, $\forall z \in \text{int}(\mathcal{Z}_1)$, $\alpha(z) = e_1$ and $\omega(z) = q_2$, $\forall z \in \text{int}(\mathcal{Z}_2)$, $\alpha(z) = e_1$ and $\omega(z) = q_3$, $\forall z \in \text{int}(\mathcal{Z}_3)$, $\alpha(z) = e_3$ and $\omega(z) = q_3$, and $\forall z \in \text{int}(\mathcal{Z}_4)$, $\alpha(z) = e_3$ and $\omega(z) = q_2$.

other hand, $\alpha(z)$ cannot be any of q_1, g, e_3 or e_1. Hence, $\alpha(z) = e_2$. Therefore, except for γ_1, every orbit in the interior of the simplex, starts from e_2 and ends at g. To sum up, we have the following

$$\forall p \in \text{int}(\Delta), \quad \alpha(p) = \begin{cases} q_1 & \text{if } p \text{ lies on the invariant unstable manifold of } q_1 \\ e_2 & \text{otherwise} \end{cases}.$$

This together with (2.4), completely determines the asymptotic behavior of any orbit in the interior of the simplex (the resulting phase portrait matches PP 15 in [24]). □

Example 2.3. Consider the following payoff matrix

$$A_3 = \begin{bmatrix} 0 & 1 & 3 \\ 3 & 0 & 1 \\ 5 & -1 & 0 \end{bmatrix}. \tag{2.5}$$

Then for (1.2), as shown in Figure 2.4(a), the fixed points are e_2 and g which are hyperbolic saddle points, e_1 and e_3 which are hyperbolic unstable, and q_3 and q_2 which are hyperbolic stable. The flow on the edges is shown in Figure 2.4(a). Similar to the previous example, since g is hyperbolic, it can be shown that no closed orbit exists. Consider the trajectories γ_1 and γ_2 that lie on the unstable invariant manifold of g. In view of Corollary 2.5, they have to converge to a fixed point. On the other hand, no trajectory converges to e_1 or e_3. So γ_1 and γ_2 converge to q_2 and q_3. If they both converge to the same $q_i, i = 2, 3$, then one of the trajectories of the stable manifold of g will be trapped in an invariant simply-connected compact set whose interior is empty of fixed points. Hence, in view of Corollary 2.4, the trajectory has to converge to either q_2 or q_3 as $t \to -\infty$, which is impossible. Hence, γ_1 and γ_2 converge to different $q_i, i = 2, 3$, e.g., γ_1 converges to q_2 and γ_2 to q_3, as shown in Figure 2.4(b). Similarly by reversing the direction of the vector field, it can be shown that one of the two trajectories of the unstable invariant manifold of g converges to e_1 and the other to e_3 as $t \to -\infty$. Then the invariant manifolds of g divide the simplex into the four areas $\mathcal{Z}_1, \ldots, \mathcal{Z}_4$. Similar to the previous examples, it can be shown that in the interior of each of the areas, every orbit starts from the stable and ends at the unstable fixed point in that area (Figure 2.4(c)). Therefore, the limit set of every point in the simplex is determined, which completes the derivation of the phase portrait (the resulting phase portrait matches PP 10 in [24]). □

2.4 Concluding remarks

For a continuously differentiable vector field, we have investigated the possible limit sets of an arbitrary point in a positively invariant compact set $\mathcal{M}$, after counting the number of its interior and boundary fixed points. As the main result, we have shown in Theorem 2.9 that when $\mathcal{M}$ contains a finite number of interior fixed points, if neither $\omega(p)$ nor $\alpha(p)$ contains just a fixed point or a closed orbit, then at least one of $\omega(p)$ or $\alpha(p)$ excludes all boundary fixed points and consists only of $\mathcal{M}$'s interior fixed points with non-fixed orbits connecting them. This tightens the classical Poincaré-Bendixson theorem by clarifying the critical role of the interior fixed points in any positively invariant compact set. Moreover, Theorem 2.9 dramatically reduces the number of possibilities for the ω limit set. For example, in the case when the interior of $\mathcal{M}$ is empty of fixed points, Theorem 2.9 reduces to Theorem 2.4 stating that the ω limit set of an arbitrary point in $\mathcal{M}$ can only be one of the fixed points on the boundary of $\mathcal{M}$. Although the state-of-the-art is still far away from extending Poincaré-Bendixson theorem to higher dimensions, it would be of great interest to develop results similar to those in this chapter for vector fields whose dimensions are greater than two at least for those $\mathcal{M}$ with special properties, e.g., when the dynamics in $\mathcal{M}$ is known not to exhibit chaos[3].

2.5 Appendix

In this Appendix, we present Fig. 2.5 which is a copy of Fig. 6 in [24]

[3]We would like to thank Dr. Hildeberto Jardon for several illuminating discussions and for introducing us to the literature, and Prof. Henk W. Broer and Prof. Robert Roussarie for their helpful comments.

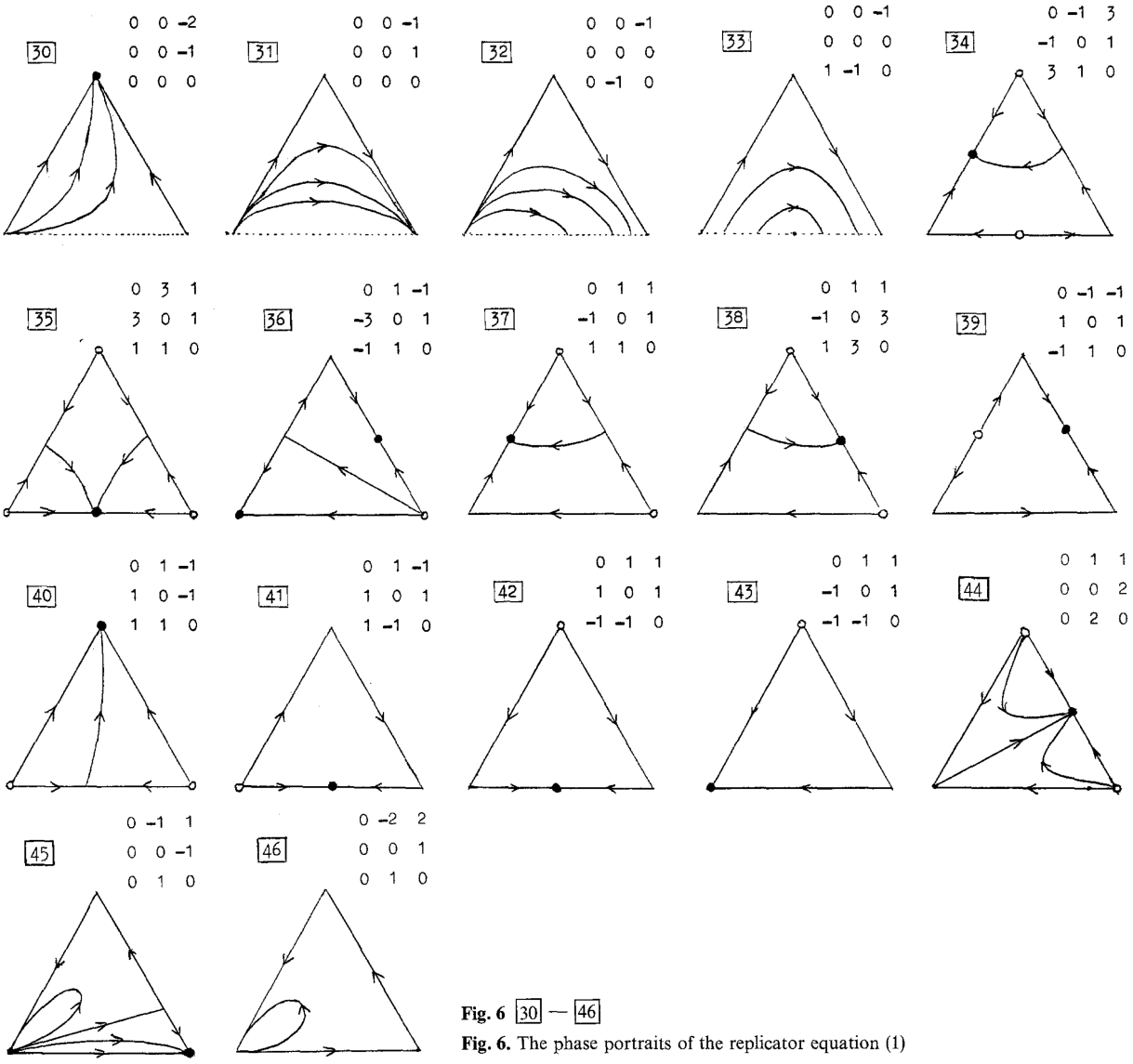


Fig. 6 30 — 46

Fig. 6. The phase portraits of the replicator equation (1)

Figure 2.5: Phase portraits 30 to 46 in [24]. Full circle • and empty circle ∘ represent a sink and a source, respectively. The figure is taken from Fig. 6 in [24].

Chapter 3

Limit sets of trajectories converging to compact curves or manifolds

For continuously differentiable vector fields, we first characterize the ω limit set of a trajectory converging to a compact curve $\Gamma \subset \mathbb{R}^n$. We show that the limit set is either a fixed point or a continuum of fixed points if Γ is a simple open curve, and otherwise can be in addition a curve in the form of either a closed orbit or a number of fixed points and compatibly oriented orbits connecting them. An implication of the result is a tightened-up version of the Poincaré-Bendixson theorem. Next we extend the results to continuously differentiable vector fields possessing an arbitrary-dimensional invariant compact manifold $\mathcal{M}$, and find the possible limit set of a trajectory converging to $\mathcal{M}$. Particularly, the area between each attracting (resp. repelling) set of the vector field restricted to $\mathcal{M}$ and the boundary of its region of attraction (resp. repulsion) does not intersect the limit set, even when the attracting and repelling sets partially coincide with the boundary of $\mathcal{M}$. This gives a sense of perspective on the extension of the Poincaré-Bendixson theorem to higher dimensions. Then we elaborate on the special case when $\mathcal{M}$ is diffeomorphic to a plane and the restriction of the vector field to $\mathcal{M}$ is hyperbolic at every fixed point. The results shed light on the qualitative properties of a vector field near a compact manifold, and are illustrated by several examples, mainly from replicator dynamics, the well-known population dynamics.

3.1 Introduction

After presenting the results on the possible limit sets of a point in a planar vector field in Chapter 2, now we broaden our attention to higher-dimensional vector fields and focus on the possible limits sets of a trajectory converging to some compact set that can be as simple as a curve or as complicated as a general invariant compact manifold. Convergence of trajectories is a fundamental problem in dynamical systems [127]. Understanding the way and to where the solutions of a system may converge provides qualitative information on the behavior of the states of a system. Most of the available research focuses on the stability of an invariant set [22, 167, 44]. Examples of fundamental results in such a setting are Lyapunov's asymptotic stability and LaSalle's invariance principle [152]. Regarding, in particular, the latter, even if it is known that a certain invariant set is, e.g. asymptotically stable, it is not straightforward to further conclude to where, precisely,

the trajectories in the neighborhood of such a set converge. The behavior of a flow near an isolated fixed point in planar dynamical systems was studied by Poincaré [132], leading to notions such as the limit set, and from there, many extensions have been worked out, e.g. the Hartman-Grobman theorem [76]. Intuitively, the limit set of a trajectory (if it exists) describes the fate of a system's state in the long run. This becomes relevant in the qualitative description of natural phenomena. For example, in population dynamics [68, 149, 138], a limit set may describe the fate of an ecosystem; in flexible robotics, convergence to the so-called slow-manifold helps to simplify the controller's design [52]. On a more mathematical side, limit sets may have a simple topological structure such as fixed points or periodic orbits, but they can also be as complicated as strange attractors. Moreover, the analysis of limit sets and convergence of trajectories becomes more complicated as the dimension of the phase space grows. The perfect example is the celebrated Poincaré-Bendixson theorem [43, 76, 184] which exists for planar systems but its extension to higher dimensions is still an open (and very hard) problem.

For spaces with the dimensions higher than one, [167] considers flows in locally compact phase space near a compact manifold $\mathcal{M}$ (see also [22]). The main result describes the stability of and convergence to $\mathcal{M}$ and provides three mutually exclusive scenarios that may occur, as explained in the following theorem. Given a point p, let $\omega(p)$ and $\alpha(p)$ denote the ω and α limit sets of p, respectively.

Theorem 3.1. *[167] Let $\mathcal{X}$ be a locally compact space and $\mathcal{M} \subset \mathcal{X}$ a non-open compact invariant set isolated from closed invariant sets. Then, one and only one of the following cases may occur.*

1. *$\mathcal{M}$ is positively asymptotically stable.*
2. *$\mathcal{M}$ is negatively asymptotically stable.*
3. *There exist points $x \notin \mathcal{M}$ and $y \notin \mathcal{M}$ such that $\emptyset \neq \omega(x) \subset \mathcal{M}$ and $\emptyset \neq \alpha(y) \subset \mathcal{M}$.*

It turns out that such a description does not hold when the phase space is a complete metric space [44]. Moreover, whenever either 1. or 3. in Theorem 3.1 occurs, it remains open to establish how the ω limit set of an arbitrary point $x \notin \mathcal{M}$ looks like. In [111], the authors describe the topological properties of asymptotically stable sets $\mathcal{M}$ within a locally compact phase space, and in [5] the authors provide a rather general description of regions of attractions of nonlinear dynamical systems. However, none of these studies treats the description of the limit set of a trajectory converging to an arbitrary-dimensional invariant compact manifold. This missing piece is needed in the analyses of various dynamical systems such as population dynamics where given that a certain species becomes extinct [135], the fate of the other species is of interest, and slow-fast systems where the dynamics close to the so-called slow manifold is relevant [75, 82, 161].

We address this open problem by investigating the actual limit set of a given trajectory converging to a compact invariant curve Γ or in general a compact invariant manifold $\mathcal{M}$ of a continuously differentiable vector field. More specifically, we are rather interested in where, within Γ or $\mathcal{M}$, the converging trajectory converges to. For the first case, we show that a trajectory converging to the curve Γ has an ω limit set in the form of a fixed point or a continuum of fixed points if Γ is a simple open curve, and otherwise can be in addition a curve in the form of either a closed orbit or a number of fixed points and compatibly oriented orbits connecting them. Part of the results will be later used in Chapter 4. For the second case, we show that the areas surrounded

by the attracting (resp. repelling) sets of the vector field restricted to $\mathcal{M}$ and the boundaries of their regions of attraction (resp. repulsion) do not attract the converging trajectory, and hence, do not intersect the corresponding limit set. The results do not impose any stability properties on $\mathcal{M}$ although the results also hold if $\mathcal{M}$ is, for example, asymptotically stable, complementing Theorem 3.1. In the special case when $\mathcal{M}$ is diffeomorphic to a plane and the restriction of the vector field to $\mathcal{M}$ is hyperbolic at every fixed point, we show that the limit set is either a fixed point or a subset of the union of the boundaries of the attracting or repelling sets of the vector field restricted to $\mathcal{M}$ and the attracting and repelling sets' attraction and repulsion regions. In general terms, the results of this chapter shed light on the qualitative properties of a vector field near invariant compact manifolds and, in particular, give a sense of perspective on the extension of the Poincaré-Bendixson theorem to higher dimensions spaces, one of the cornerstones of the study of dynamical systems.

The rest of the chapter is arranged as follows. In Section 3.2, we provide definitions and preliminary results. In Section 3.3, we present the main results for the case when a trajectory converges to a compact curve. As a consequence, we are able to give a tightened up version of Poincaré-Bendixson theorem, see Theorem 3.4. We proceed in Section 3.4 with a couple of nontrivial academic examples where the main results are applied to. In Section 3.5, we deal with the problem of convergence to an m-dimensional, $m \in \mathbb{N}$, compact invariant set. We construct a compact invariant subset $\mathcal{N} \subset \mathcal{M}$ which contains all the possible limit sets of a trajectory converging to $\mathcal{M}$. Then, in Section 3.6, we discuss the interesting case when the manifold $\mathcal{M}$ is 2-dimensional and has only hyperbolic singularities. Next, in Section 3.7 we provide several examples for our main results, particularly some in the context of population dynamics. We finalize the chapter with the concluding remarks.

3.2 Preliminaries

We consider a $\mathbf{C}^\ell$, $\ell \geq 1$, vector field

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n. \tag{3.1}$$

The following is a fundamental result on the continuity of the vector field with respect to the initial condition.

Theorem 3.2. *[78, Theorem 3.5] Let $f(x)$ be continuous and locally Lipschitz in x on $\mathcal{X}$ where $\mathcal{X} \subset \mathbb{R}^n$ is an open connected set. Let $y(t)$ be a solution of $\dot{x} = f(x)$ with $y(t_0) = y_0 \in \mathcal{X}$. Suppose $y(t)$ is defined and belongs to $\mathcal{X}$ for all $t \in [t_0, t_1]$. Then given $\epsilon > 0$, there is $\delta > 0$ such that if*

$$\|z_0 - y_0\| < \delta,$$

then there is a unique solution $z(t)$ of $\dot{x} = f(x)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and $z(t)$ satisfies

$$\|z(t) - y(t)\| < \epsilon \qquad \forall t \in [t_0, t_1].$$

The boundary and interior of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted by $\mathrm{bd}(\mathcal{S})$ and $\mathrm{int}(\mathcal{S})$, respectively. The following definitions are essential for stability and attractiveness notions.

Definition 3.1 (Trapping region [103]). *A set $\mathcal{U} \subset \mathbb{R}^n$ is a* trapping region *(of the vector field f) if it is compact and $\varphi_t(\mathcal{U}) \subset \text{int}(\mathcal{U})$ for all $t > 0$.*

Note that, by definition, a trapping region 'shrinks' as it evolves over time.

Definition 3.2 (Attracting set [103]). *A set $\mathcal{A}$ is an* attracting set *(of the vector field f) if there is a trapping region $\mathcal{U}$ so that $\mathcal{A} \subset \mathcal{U}$ and*

$$\bigcap_{t>0} \varphi_t(\mathcal{U}) = \mathcal{A}.$$

The set $\mathcal{U}$ is called a trapping region of $\mathcal{A}$.

An invariant set $\mathcal{S}$ is said to be *stable* if for any neighborhood $\mathcal{U}$ of $\mathcal{S}$, there is a neighborhood $\mathcal{V}$ of $\mathcal{S}$ such that $\phi_t(\mathcal{V}) \subseteq \mathcal{U}$ for all $t > 0$, that is, all trajectories with initial conditions in $\mathcal{V}$ stay in $\mathcal{U}$. An invariant set $\mathcal{S}$ is said to be *asymptotically stable* if it is stable and for every $x \in \mathcal{V}$, $d(\varphi_t(x), \mathcal{S}) \to 0$ as $t \to \infty$ [103]. The following result reveals the relationship between the previous definitions.

Lemma 3.1. *[103, Lemma 4.18] An attracting set is asymptotically stable. Conversely if a compact set is asymptotically stable, then it is an attracting set.*

In a similar way, a set $\mathcal{R} \subset \mathbb{R}^n$ is a *repelling set* if it is an attracting set when time is reversed.

Definition 3.3 (Attraction region). *The* attraction region *of an attracting set $\mathcal{A}$, denoted by $\Omega(\mathcal{A})$, is defined by*

$$\Omega(\mathcal{A}) = \{x \in \mathbb{R}^n \mid d\left(\varphi_t(x), \mathcal{A}\right) \to 0 \text{ as } t \to \infty\}.$$

Note that $\Omega(\mathcal{A})$ consists of orbits. Therefore, the attraction region can be equivalently represented by

$$\Omega(\mathcal{A}) = \bigcup_{t \leq 0} \varphi_t(\mathcal{U}) \tag{3.2}$$

where $\mathcal{U}$ is any trapping region of $\mathcal{A}$ [103]. Note that $\Omega(\mathcal{A})$ is open, and any trapping region $\mathcal{U} \supset \mathcal{A}$ is a compact and proper subset of $\Omega(\mathcal{A})$. The *repulsion region* of a repelling set $\mathcal{R}$, denoted by $\Omega(\mathcal{R})$, is defined similarly, but when $t \to -\infty$.

Given a compact invariant set $\mathcal{M} \subset \mathbb{R}^n$, the restriction of the vector field f on $\mathcal{M}$ is denoted by $f|_{\mathcal{M}}$. A set $\mathcal{A} \subseteq \mathcal{M}$ is said to be an *attracting (resp. repelling) set of f in $\mathcal{M}$* if $\mathcal{A}$ is an attracting (resp. repelling) set of the restriction $f|_{\mathcal{M}}$. Correspondingly, the attraction region $\Omega_{\mathcal{M}}(\mathcal{A})$ and repelling region $\Omega_{\mathcal{M}}(\mathcal{R})$ are defined on $\mathcal{M}$ with respect to $f|_{\mathcal{M}}$.

The definition of an attracting set of f in $\mathcal{M}$ readily implies that $\mathcal{A}$ is in the interior of some trapping region $\mathcal{U}$, i.e., $\mathcal{A} \subset \text{int}(\mathcal{U})$, yielding $\mathcal{A} \subset \text{int}(\mathcal{M})$. However, as we will show in Section 3.7, there are cases where there exists a set $\mathcal{A}'$ contained in or intersecting the boundary of $\mathcal{M}$, which all trajectories in a neighborhood of $\mathcal{A}'$ in $\mathcal{M}$, are attracted to. Such a set $\mathcal{A}'$ is not an attracting set according to Definition 3.2, yet it behaves similar to one. This motivates the following definitions.

Definition 3.4 (Semi-Trapping region). *Given an invariant compact set $\mathcal{M} \subset \mathbb{R}^n$, a set $\mathcal{U}' \subset \mathcal{M}$ is a* semi-trapping region *of f in $\mathcal{M}$, if it is compact, $\mathcal{U}' \cap \text{bd}(\mathcal{M}) \neq \emptyset$, and the set $\hat{\mathcal{U}} = \mathcal{U}' \cap \text{int}(\mathcal{M})$ satisfies $\varphi_t(\hat{\mathcal{U}}) \subset \text{int}(\hat{\mathcal{U}})$ for all $t > 0$.*

Definition 3.5 (Semi-Attracting set)**.** *Given an invariant compact set $\mathcal{M} \subset \mathbb{R}^n$, a set $\mathcal{A}' \subset \mathcal{M}$ is a* semi-attracting set *of the vector field $f|_{\mathcal{M}}$, if $A' \cap \mathrm{bd}(\mathcal{M}) \neq \emptyset$, and there is a semi-trapping region $\mathcal{U}'$ so that $\mathcal{A}' \subset \mathcal{U}'$ and*

$$\bigcap_{t>0} \varphi_t(\mathcal{U}') = \mathcal{A}'.$$

The set $\mathcal{U}'$ is called a semi-trapping region of $\mathcal{A}'$.

3.3 Limit set of trajectories converging to curves

Consider a $\mathbf{C}^\ell$, $\ell \geq 1$, vector field (3.1). Let Γ be a compact invariant curve under the vector field, and consider a point $p \in \mathbb{R}^n$, the trajectory through which converges to Γ as $t \to \infty$. Our goal is to find the possibilities for $\omega(p)$. One of the trivial candidates is the set of fixed points on Γ, denoted by $\mathcal{N}$. If $\mathcal{N} = \emptyset$, then Γ is a closed orbit. If $\mathcal{N} \neq \emptyset$, then $\mathcal{N}$ is the union of its connected components $\mathcal{N}_i$, each of which we refer to as a *fixed point component of* Γ and is either a single fixed point or a continuum of fixed points. For the rest of Γ, denoted by $\mathcal{I} = \Gamma - \mathcal{N}$, we observe that $\mathcal{I}$ is either empty or the union of the non-fixed orbits in Γ, denoted by $\mathcal{I}_i$. Each $\mathcal{I}_i$ is connected and disjoint from any other $\mathcal{I}_j$, i.e., $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ for all $i \neq j$. We show in Proposition 3.1 that the non-fixed orbits $\mathcal{I}_i$ do not intersect $\omega(p)$, unless they are part of a flow-aligned curve defined in the following.

Definition 3.6 (Flow-aligned curves)**.** *We say an invariant simple closed curve Γ is a* simple flow aligned *curve, if it is either a closed orbit or a continuum of fixed points, or each $\mathcal{N}_i \subseteq \Gamma$ contains the ω limit point of exactly one non-fixed orbit and the α limit point of exactly one (not necessarily different) non-fixed orbit in Γ. A connected union of simple flow-aligned curves is called a* flow-aligned curve.

The last case of the definition of a simple flow-aligned curve implies the union of sets of fixed points (either a single or a continuum of fixed points) and compatibly oriented orbits joining them.

Proposition 3.1. *Consider the vector field* (3.1)*, and let p be an arbitrary point in $\mathbb{R}^n$. If $\varphi_t(p)$ converges to a compact curve Γ, then for every non-fixed orbit $\mathcal{I}_i \subseteq \Gamma$ that is not a subset of any flow-aligned simple sub-curve of Γ, it holds that $\omega(p) \cap \mathcal{I}_i = \emptyset$.*

Proposition 3.1 implies that if a trajectory converges to an invariant curve having non-fixed orbits, then the corresponding ω limit set is built upon flow-aligned curves. We remark, however, that the converse is not necessarily true. Using saddle points, one can easily construct an invariant flow-aligned curve that is not an ω limit set of any point not belonging to the curve.

For the proof of Proposition 3.1, we first consider a non-fixed orbit $\mathcal{I}_i \subseteq \Gamma$ that is not a subset of any flow-aligned simple sub-curve of Γ. It follows that $\mathcal{I}_i$ is not a closed orbit, and hence is diffeomorphic to a line segment since it is compact. So, without loss of generality, we assume $\mathcal{I}_i$ is a line segment. Next, let $(z_1, \ldots, z_n)$ be a coordinate system such that $\mathcal{I}_i$ lies along the z_1-axis where one ending point of $\mathcal{I}_i$ coincides with the origin and the other, denoted by r, lies on the positive z_1-axis. Since $\mathcal{I}_i$ is an orbit and hence invariant, there exists a flow on $\mathcal{I}_i$, say in the positive direction of the z_1-axis. We show that this flow does not allow any point on $\mathcal{I}_i$ to be an

ω limit point of p. Consider a point $q \in \mathcal{I}_i$ and constant $\epsilon \in (q_1, r_1 - q_1)$ where x_j denotes the jth entry of x. Define $\mathcal{I}_i^\epsilon(q) \subset \mathcal{I}_i$ as a line segment of length 2ϵ, centered at q, i.e.,

$$\mathcal{I}_i^\epsilon(q) = \{\zeta \in \mathcal{I}_i \,|\, q_1 - \epsilon \leq \zeta_1 \leq q_1 + \epsilon\}\,.$$

We show the existence of a sufficiently small tubular neighborhood of $\mathcal{I}_i^\epsilon(q)$ in which the flow in the z_1 direction is always positive. Denote the z-coordinate of $\varphi_t(p)$ by $z(t)$. For $\delta > 0$, define $\mathcal{Q}^\delta$, the n-dimensional tubular neighborhood of $\mathcal{I}_i^\epsilon(q)$, in the z-coordinate by

$$\mathcal{Q}^\delta = \left\{ z \in \mathbb{R}^n \,\middle|\, z_1 \in \mathcal{I}_i^\epsilon(q),\ d(z, \mathcal{I}_i^\epsilon(q)) \leq \delta \right\}.$$

Lemma 3.2. *There exist constants $\delta > 0$ and $r > 0$ such that if $z \in \mathcal{Q}^\delta$, then $\dot{z}_1 > r > 0$.*

Proof. The proof follows from the flow-box or rectification theorem [15, Corollary 10 of Ch. 2]. □

Proof of Proposition 3.1. Take δ from Lemma 3.2 and define the following two subsets of the boundary of $\mathcal{Q}^\delta$: $\mathcal{D}^1 = \{z \in \mathcal{Q}^\delta \,|\, z_1 = q_1 - \epsilon\}$ and $\mathcal{D}^2 = \{z \in \mathcal{Q}^\delta \,|\, z_1 = q_1 + \epsilon\}$. If there exists some point $\xi \in \omega(p) \cap \mathcal{I}_i$, then there exists a time sequence $\{t_j\}_{j=1}^\infty, t_j \to \infty$, such that $\varphi(t_j, p) \in \mathcal{Q}^\delta$ for all t_j. On the other hand, in view of Lemma 3.2, corresponding to each t_j there exists some $\hat{t}_j > t_j$ such that $\varphi(\hat{t}_j, p) \notin \mathcal{Q}^\delta$. So, if $q \in \omega(p)$, it would mean that $\varphi_t(p)$ must enter $\mathcal{Q}^\delta$ infinitely many times. However, the fact that $\varphi_t(p) \to \Gamma$ as $t \to \infty$ implies the existence of $M > 0$ such that

$$d\left(\varphi_t(p), \Gamma\right) < \delta \quad \forall t > M. \tag{3.3}$$

In other words, after $t = M$, the flow through p stays in a distance of δ from Γ. Moreover, in view of Lemma 3.2, if $\varphi_t(p)$ enters $\mathcal{Q}^\delta$, it enters from $\mathcal{D}^1$ and leaves from $\mathcal{D}^2$ in finite time. Hence, for $\varphi_t(p)$ to enter $\mathcal{Q}^\delta$ for the second time after $t = M$, it must get from $\mathcal{D}^2$ to $\mathcal{D}^1$ in a distance δ of Γ without entering $\mathcal{Q}^\delta$. This implies the existence of a simple closed sub-curve $\Gamma' \subseteq \Gamma$ connecting $\mathcal{D}^2$ to $\mathcal{D}^1$, containing $\mathcal{I}_i$. On the other hand, if none of such sub-curves are flow aligned, then again due to Lemma 3.2, it is impossible for $\varphi_t(p)$ to return to $\mathcal{Q}^\delta$, leading to the proof. □

Proposition 3.1 results in the following two corollaries.

Corollary 3.1. *Consider the vector field* (3.1)*, and let p be an arbitrary point in $\mathbb{R}^n$. If $\varphi_t(p)$ converges to a compact simple open curve Γ, then $\omega(p)$ is a single fixed point or a continuum of fixed points on Γ.*

Proof. First we look at the case when Γ is invariant. From Proposition 3.1 and since Γ is simple, $\omega(p) \cap \mathcal{I} = \emptyset$. Hence, $\omega(p) \subseteq \Gamma - \mathcal{I} = \mathcal{N}$. Moreover, it follows from Lemma 1.1 that $\omega(p)$ is connected since Γ is compact. Hence, there exists a fixed point component $\mathcal{N}_i \subseteq \mathcal{N}$ such that $\omega(p) \subseteq \mathcal{N}_i$, leading to the result. Now if Γ is not invariant, from Lemma 1.1 we know that $\omega(p)$ is invariant; hence, $\omega(p)$ is a subset of the invariant part of Γ. Then by repeating the same argument as above, we reach the conclusion. □

As an immediate consequence of Corollary 3.1, if the vector field does not admit a continuum of fixed points and Γ is a simple open curve, then the trajectory through p converges to a fixed point on Γ.

Corollary 3.2. *Consider the vector field* (3.1)*, and let p be an arbitrary point in $\mathbb{R}^n$. If $\varphi_t(p)$ converges to a simple closed curve Γ, then one of the following holds for $\omega(p)$:*

- *if Γ is flow-aligned, then $\omega(p)$ is a single fixed point on Γ, a continuum of fixed points on Γ or the whole Γ itself;*
- *otherwise, $\omega(p)$ is either a single fixed point or a continuum of fixed points on Γ.*

Proof. If Γ is not invariant, then as stated in the proof of Corollary 3.1, $\varphi_t(p)$ converges to the invariant part of Γ which is a simple open curve, and hence, not flow-aligned. The result then follows from Corollary 3.1. So consider the case when Γ is invariant. Since Γ is compact, it follows from Lemma 1.1 that $\omega(p)$ is connected. Having this in mind, first we consider the case when Γ is flow-aligned. If $\omega(p) = \Gamma$, the result is trivial; otherwise, since $\omega(p)$ is connected, it is a subset of a simple open sub-curve Γ' of Γ. Then $\varphi_t(p)$ converges to Γ', which in view of Corollary 3.1 results in $\omega(p)$ being a fixed point or a continuum of fixed points on Γ. The case when Γ is not flow-aligned follows from Proposition 3.1 and the proof is similar to the proof of Corollary 3.1. □

Now we proceed to the main result of this chapter where a (possibly non-simple, open or closed) compact curve in general, is considered as the curve to which $\varphi_t(p)$ converges.

Theorem 3.3. *Consider the vector field* (3.1)*, and let p be an arbitrary point in $\mathbb{R}^n$. If $\varphi_t(p)$ converges to a compact curve $\Gamma \subset \mathbb{R}^n$. Then $\omega(p)$ is one of the following sets*

- *a single fixed point on Γ,*
- *a continuum of fixed points on Γ, or*
- *a flow-aligned sub-curve of Γ and possibly continuum of fixed points connected to it.*

Proof. We prove by contradiction: if $\omega(p)$ is not in the form of any of the three cases mentioned in the theorem, then it contains a non-fixed orbit $\mathcal{I}_i \subseteq \Gamma$ that is not a subset of any flow-aligned closed sub-curve of Γ, which is a contradiction according to Proposition 3.1. □

Theorem 3.3 clarifies the flow structure of the ω limit sets within a curve. This particularly finds extensive application in planar dynamics, where limit sets in invariant regions are always either a curve or a fixed point. After all, the third case in Poincaré-Bendixson theorem leaves many possibilities for the geometrical structure of the limit set, a wide range of which is impossible based on Theorem 3.3. This motivates us to present the following tightened-up version of Poincaré-Bendixson theorem. We note that a similar result, but with a different proof, has been stated in [12, Theorem 68].

Theorem 3.4 (Tightened-up Poincaré-Bendixson Theorem)**.** *Consider a $\mathbf{C}^\ell, \ell \geq 1$, planar vector field. Let $\mathcal{M}$ be a positively invariant compact set for the vector field, and assume that $\mathcal{M}$ has a finite number of fixed points. Let $p \in \mathcal{M}$, and consider $\omega(p)$. Then one of the following holds:*

1. *$\omega(p)$ is a singleton;*
2. *$\omega(p)$ is a closed orbit;*

3. *$\omega(p)$ is a flow-aligned curve consisting of a finite number of fixed points $p_1, \ldots, p_n$ and orbits γ with $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$, where p_i and p_j are not necessarily different.*

Proof. The proof follows Theorems 2.2 and 3.3 and the fact that $\mathcal{M}$ does not possess any continuum of fixed points. □

3.4 Examples

Example 3.1. This example is related to Poincaré-Bendixson theorem. In Figure 3.1 we present three invariant sets that, according to Theorem 2.2, are possible ω limit sets for a planar vector field. However, from Theorem 3.4, we know that they, in fact, are not.

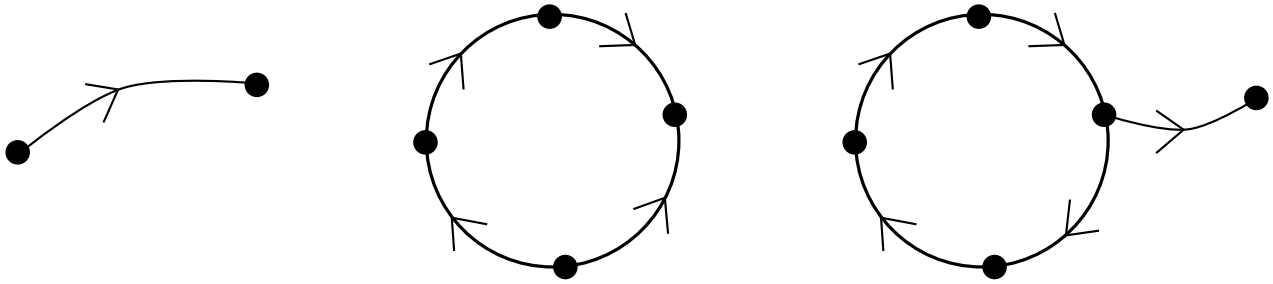


Figure 3.1: According to Theorem 2.2, all of the shown sets could possibly be ω limit sets of a planar vector field. However, with the refinement of Theorem 3.4, we rule out such cases.

Example 3.2. Consider the $\mathbf{C}^1$ planar vector field

$$\dot{r} = -(r-1)^5, \qquad \dot{\theta} = \begin{cases} 0 & r = 1 \\ -(r-1)^3 \sin\left(\frac{3}{2(r-1)}\right) & r \neq 1 \end{cases}, \tag{3.4}$$

possessing a continuum of fixed points on the unit circle, denoted by Γ. If for some point p in the plane, $\varphi_t(p)$ converges to Γ, then Theorem 3.3 says that $\varphi_t(p)$ either converges to a fixed point or continuum of fixed points in Γ. Although seemingly counter intuitive, the continuum does not need to be the whole Γ, and can be only a subset of it as shown in Figure 3.2.

3.5 Limit set of trajectories converging to compact invariant manifolds

Consider the $\mathbf{C}^\ell$, $\ell \geq 1$, vector field (3.1). We assume that there is an m-dimensional, $m \in \{1, \ldots, n\}$, compact manifold $\mathcal{M} \subset \mathbb{R}^n$ invariant under the flow of f. Denote by $\mathscr{A}$ the set of all attracting sets of f in $\mathcal{M}$, that is

$$\mathscr{A} = \{\mathcal{A} \subset \mathcal{M} \,|\, \mathcal{A} \text{ is an attracting set of } f \text{ in } \mathcal{M}\}\,.$$

Corresponding to each $\mathcal{A} \in \mathscr{A}$, the set $\mathcal{U}_\mathcal{A} := \Omega_\mathcal{M}(\mathcal{A}) - \mathcal{A}$ does not contain any invariant subset, and every trajectory in $\mathcal{U}_\mathcal{A}$ converges to $\mathcal{A}$. Hence, intuitively, $\mathcal{U}_\mathcal{A}$ does not attract any

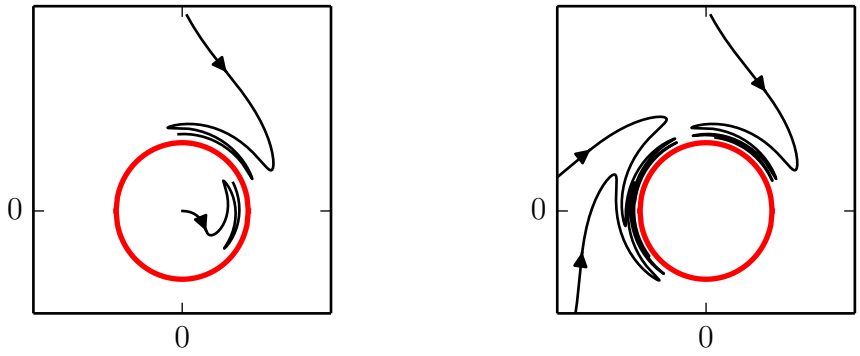


Figure 3.2: Phase portraits of (3.4). The red unit circle, Γ, consists only of fixed points. Depending on the initial condition, a trajectory of (3.4) approaching Γ converges to a proper subset of Γ.

trajectory converging to $\mathcal{M}$. The same holds for each $\mathcal{U}_\mathcal{R} := \Omega_\mathcal{M}(\mathcal{R}) - \mathcal{R}$ corresponding to an $\mathcal{R} \in \mathscr{R}$ where $\mathscr{R}$ is defined similar to $\mathscr{A}$, but for the repellors $\mathcal{R}$ of f in $\mathcal{M}$ (see Fig. 3.3-(a) and 3.3-(b)). Therefore, the set

$$\mathcal{N}_{\mathscr{A},\mathscr{R}} = \mathcal{M} - \bigcup_{\mathcal{A}\in\mathscr{A}} \mathcal{U}_\mathcal{A} - \bigcup_{\mathcal{R}\in\mathscr{R}} \mathcal{U}_\mathcal{R}$$

is a candidate for containing the limit set of a trajectory converging to $\mathcal{M}$ (see Fig. 3.3-(c)). This is upheld by the fact that $\mathcal{N}_{\mathscr{A},\mathscr{R}}$ does not contain any trapping region (both in forward and backward time) and that among its elements, $\mathcal{N}_{\mathscr{A},\mathscr{R}}$ contains every fixed point, attracting set and repelling set of $\mathcal{M}$.

Similar arguments hold for the semi-attracting regions of $\mathcal{M}$. Just as before, define the set of all semi-attracting sets of f in $\mathcal{M}$ by

$$\mathscr{A}' = \{\mathcal{A}' \subset \mathcal{M} \,|\, \mathcal{A}' \text{ is a semi-attracting set of } f \text{ in } \mathcal{M}\}$$

and the region of attraction of each $\mathcal{A}' \in \mathscr{A}'$ by

$$\Omega_\mathcal{M}(\mathcal{A}') = \{x \in \mathcal{M} \,|\, d(\phi_t(x), \mathcal{A}') \to 0 \text{ as } t \to \infty\}\,.$$

Then, again, for every $\mathcal{A}' \in \mathscr{A}'$, the set $\mathcal{U}_{\mathcal{A}'} := \Omega_\mathcal{M}(\mathcal{A}') - \mathcal{A}'$, intuitively does not attract any trajectory converging to $\mathcal{M}$. This time, however, $\mathcal{U}_{\mathcal{A}'}$ contains some part of the boundary of $\mathcal{M}$, which is due to the fact that $\mathcal{A}' \cap \mathrm{bd}(\mathcal{M}) \neq \emptyset$ and $\mathrm{bd}(\mathcal{M})$ is invariant, yielding $\Omega_\mathcal{M}(\mathcal{A}') \cap \mathrm{bd}(\mathcal{M}) \neq \emptyset$ (see Fig. 3.4-(a) to (b)). The same holds for each $\mathcal{U}_{\mathcal{R}'} := \Omega_\mathcal{M}(\mathcal{R}') - \mathcal{R}'$ corresponding to a $\mathcal{R}' \in \mathscr{R}'$ where $\Omega_\mathcal{M}(\mathcal{R}')$ and $\mathscr{R}'$ are defined similarly to $\Omega_\mathcal{M}(\mathcal{A}')$ and $\mathscr{A}'$. Now by subtracting these two types of sets from $\mathcal{N}_{\mathscr{A}',\mathscr{R}'}$, we obtain the following candidate for the limit sets of trajectories converging to $\mathcal{M}$:

$$\mathcal{N} = \mathcal{M} - \bigcup_{\mathcal{A}\in\mathscr{A}} \mathcal{U}_\mathcal{A} - \bigcup_{\mathcal{R}\in\mathscr{R}} \mathcal{U}_\mathcal{R} - \bigcup_{\mathcal{A}'\in\mathscr{A}'} \mathcal{U}_{\mathcal{A}'} - \bigcup_{\mathcal{R}'\in\mathscr{R}'} \mathcal{U}_{\mathcal{R}'}. \tag{3.5}$$

Since the limit set of a trajectory converging to $\mathcal{M}$ is connected, the trajectory may converge only to a connected component of $\mathcal{N}$ which we therefore call a *candidate limit set* and denote it by $\mathcal{N}_i$. Then $\mathcal{N} = \bigcup_i \mathcal{N}_i$ where each $\mathcal{N}_i$ is a maximal connected subset of $\mathcal{N}$ and $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ for all $i \neq j$ (see Fig. 3.4-(e)). The following result reveals the invariance of the candidate limit sets under the vector field, implying in particular the invariance of their boundaries.

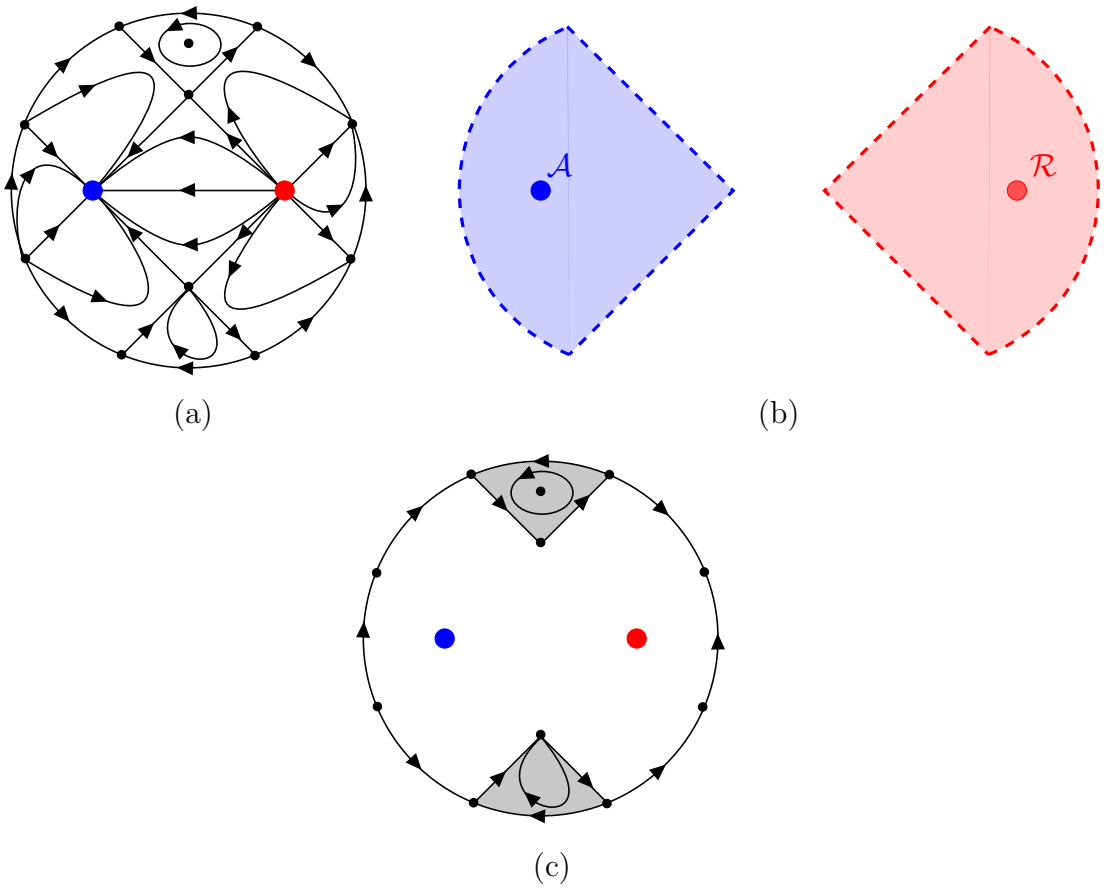


Figure 3.3: (a) An example of $\mathcal{M}$ as an invariant set in the form of a disc. The set $\mathcal{M}$ has 1 attractor (a stable fixed point in blue) $\mathcal{A}$ and 1 repellor (an unstable fixed point in red) $\mathcal{R}$. (b) The regions of attraction $\Omega_{\mathcal{M}}(\mathcal{A})$ and repulsion $\Omega_{\mathcal{M}}(\mathcal{R})$ are open sets depicted in light-blue and light-red respectively. (c) The set $\mathcal{N}_{\mathscr{A}',\mathscr{R}'} = \mathcal{M} - \mathcal{U}_{\mathcal{A}} - \mathcal{U}_{\mathcal{R}}$.

Proposition 3.2. *Each candidate limit set $\mathcal{N}_i$ is invariant.*

We need the following lemma for the proof.

Lemma 3.3. *Under the vector field* (3.1)*, if a set is invariant, so is its boundary.*

Proof. Following a similar proof to that of Lemma 9 in [137], we prove by contradiction. Consider some set $\mathcal{S}$ that is invariant under (3.1), and assume on the contrary that $\mathrm{bd}(\mathcal{S})$ is not invariant. Then there exists a point $s \in \mathrm{bd}(\mathcal{S})$ such that for some time $T > 0$, either i) $\varphi_T(s) \in \mathrm{int}(\mathcal{S})$ if $s \in \mathcal{S}$ or ii) $\varphi_T(s)$ is bounded away from $\mathcal{S}$ if $s \notin \mathcal{S}$. Let $d = d(\varphi_T(s), \mathrm{bd}(\mathcal{S}))$ where $d > 0$. It then follows from the continuity of the vector field with respect to the initial conditions (Theorem 3.2) and the fact that s is a boundary point that in Case i) there is a point $y \notin \mathcal{S}$ that is close enough to s such that $\varphi_T(y)$ belongs to $\mathcal{S}$ and is at least $\frac{d}{2}$ away from $\mathrm{bd}(\mathcal{S})$, implying $\varphi_T(y) \in \mathrm{int}(\mathcal{S})$, and in Case ii) there is a point $y \in \mathcal{S}$ that is close enough to s such that $\varphi_T(y)$ is at least $\frac{d}{2}$ away from $\mathcal{S}$. Both cases contradict the fact that $\mathcal{S}$ is invariant. □

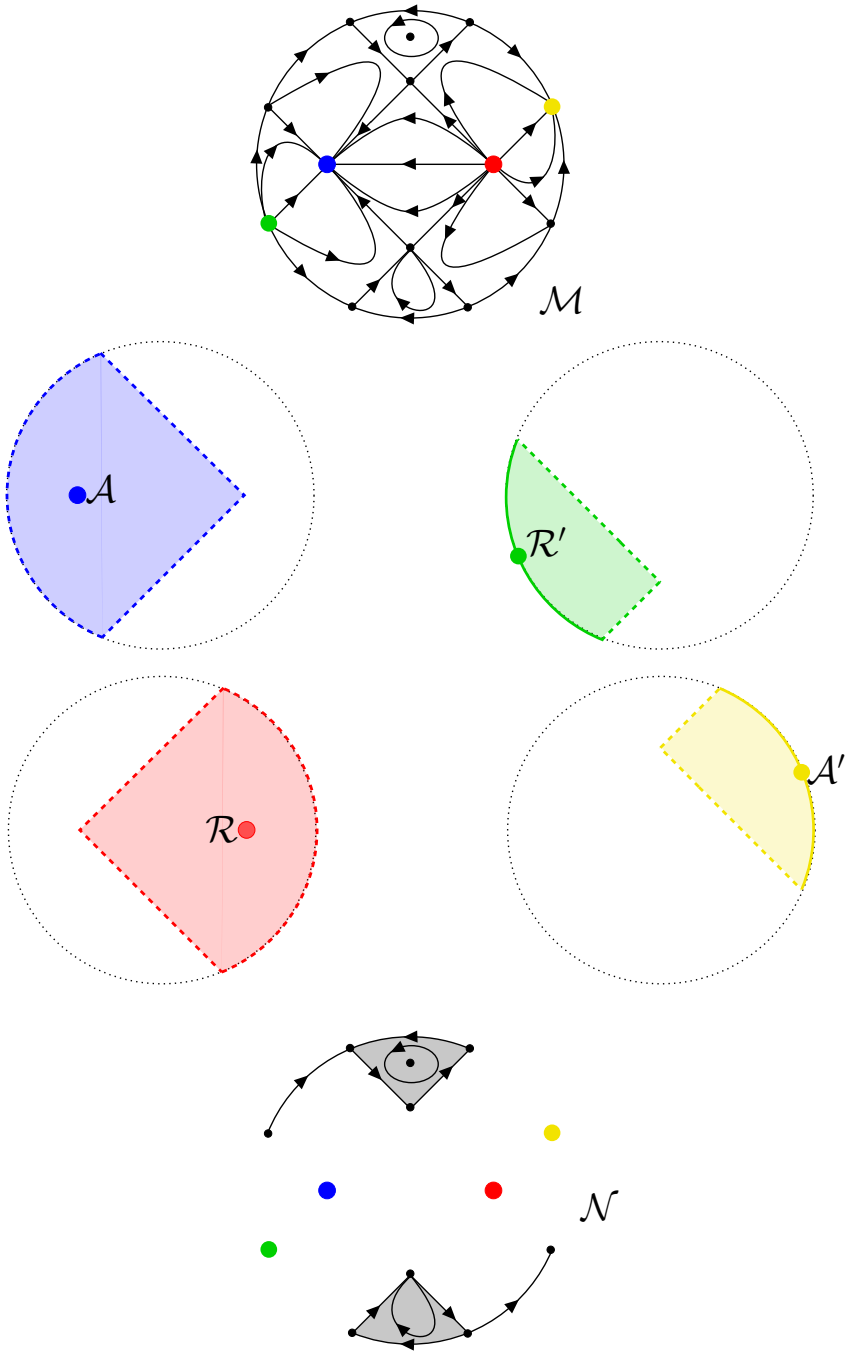


Figure 3.4: An example of $\mathcal{M}$ as an invariant set in the form of a disc. The set $\mathcal{M}$ has 1 attracting set $\mathcal{A}$ (a stable fixed point in blue), 1 repelling set $\mathcal{R}$ (an unstable fixed point in red), 1 semi-attracting set $\mathcal{A}'$ (in yellow), 1 semi-repelling set $\mathcal{R}'$ (in green). (b) $\Omega_{\mathcal{M}}(\mathcal{A})$ and $\Omega_{\mathcal{M}}(\mathcal{R})$ are open sets depicted in light-blue and light-red; $\Omega_{\mathcal{M}}(\mathcal{A}')$ and $\Omega_{\mathcal{M}}(\mathcal{R}')$ are the sets depicted in light-green and light-yellow respectively. (c) The set $\mathcal{N} = \mathcal{M} - \mathcal{U}_{\mathcal{A}} - \mathcal{U}_{\mathcal{R}} - \mathcal{U}_{\mathcal{A}'} - \mathcal{U}_{\mathcal{R}'}$.

Proof of Proposition 3.2. Consider $\mathcal{N}$ defined in (3.5). The boundary of each $\mathcal{U}_{\mathcal{A}}$ is the union of the boundary of the attraction region $\Omega_{\mathcal{M}}(\mathcal{A})$ and the boundary of the attracting set $\mathcal{A}$, respectively, which are both invariant. The same holds for every $\mathcal{U}_{\mathcal{R}}$,$\mathcal{U}_{\mathcal{A}'}$ and $\mathcal{U}_{\mathcal{R}'}$. On the other hand, $\mathcal{M}$ is invariant, implying that its boundary is also invariant in view of Lemma 3.3. Hence, because of the definition (3.5), $\mathcal{N}$ is invariant. Therefore, each $\mathcal{N}_i$ is invariant since their union constructs $\mathcal{N}$ and they are mutually disjoint. □

Therefore, the boundary of each $\mathcal{N}_i$ is also invariant, implying the following.

Corollary 3.3. *The boundary of each candidate limit set $\mathcal{N}_i$ is a union of orbits.*

Proof. The proof follows Proposition 3.2 and Lemma 3.3. □

As the main result of this section, we show that the $\mathcal{N}_i$'s are the only possible sets to where a trajectory converging to $\mathcal{M}$ may eventually converge. So the limit set of such a converging trajectory is contained in one of the $\mathcal{N}_i$'s, i.e., $\exists \mathcal{N}_i : \omega(p) \subseteq \mathcal{N}_i$.

Theorem 3.5. *Consider the vector field* (3.1) *possessing an invariant compact manifold $\mathcal{M} \subset \mathbb{R}^n$. Let p be an arbitrary point in $\mathbb{R}^n$. If $\varphi_t(p) \to \mathcal{M}$ as $t \to \infty$, then there exists a candidate limit set $\mathcal{N}_i$ such that $\varphi_t(p) \to \mathcal{N}_i$ as $t \to \infty$.*

To get intuition on the theorem, consider a point p belonging to the invariant set $\mathcal{M}$, which obviously satisfies the condition $\varphi_t(p) \to \mathcal{M}$ as $t \to \infty$. Then either there exists some $\mathcal{N}_i \ni p$ or p belongs to some $\mathcal{U}_{\mathcal{A}}$,$\mathcal{U}_{\mathcal{R}}$,$\mathcal{U}_{\mathcal{A}'}$ or $\mathcal{U}_{\mathcal{R}'}$. The first case readily yields the result. For the second, we note that no point q in any of $\mathcal{U}_{\mathcal{A}}$,$\mathcal{U}_{\mathcal{R}}$,$\mathcal{U}_{\mathcal{A}'}$ or $\mathcal{U}_{\mathcal{R}'}$ can be an ω limit point of p since due to the openness of these sets, any sufficiently small neighborhood of q will eventually converge to $\mathcal{A}, \Omega_{\mathcal{R}}, \mathcal{A}'$ and $\Omega_{\mathcal{R}'}$, respectively, making q a wandering point. So the intersection of $\omega(p)$ with all $\mathcal{U}_{\mathcal{A}}$,$\mathcal{U}_{\mathcal{R}}$,$\mathcal{U}_{\mathcal{A}'}$ and $\mathcal{U}_{\mathcal{R}'}$ is empty. Hence, $\omega(p)$ is a subset of $\mathcal{M}$ minus all $\mathcal{U}_{\mathcal{A}}$,$\mathcal{U}_{\mathcal{R}}$,$\mathcal{U}_{\mathcal{A}'}$ and $\mathcal{U}_{\mathcal{R}'}$, which results in $\omega(p) \subseteq \mathcal{N}$. Then due to the connectedness of $\omega(p)$, there exists some $\mathcal{N}_i \subseteq \mathcal{N}$ such that $\omega(p) \subseteq \mathcal{N}_i$, leading to the result.

To rigorously prove Theorem 3.5, in general, for p in or outside of $\mathcal{M}$, we first focus on an arbitrary point q in a $\mathcal{U}_{\mathcal{A}}$ corresponding to some $\mathcal{A} \in \mathscr{A}$, and show that q cannot be an ω limit point of p. For this, we construct $\mathcal{B}_\delta(q)$, a neighborhood of q in $\mathbb{R}^n$, and show the existence of some finite time, after which $\varphi_t(p)$ never intersects $\mathcal{B}_\delta(q)$. To construct $\mathcal{B}_\delta(q)$, we first show that q lies in some trapping region of $\mathcal{A}$ and then expand this trapping region under the flow for some finite time in both positive and negative direction to get a neighborhood of q in $\mathcal{M}$, denoted by $\mathcal{B}_0(q)$. The neighborhood $\mathcal{B}_0(q)$ is then expanded to $\mathcal{B}_\delta(q)$ by further including all points in $\mathbb{R}^n$ with the distance δ from $\mathcal{B}_0(q)$, and whose projection onto $\mathcal{M}$ lies in $\mathcal{B}_0(q)$. We formally construct $\mathcal{B}_\delta(q)$ by the following steps.

First, we prove that trapping regions remain so under the flow. Given a set $\mathcal{S}$, we define $\mathcal{S}^t = \varphi_t(\mathcal{S})$ for $t \in \mathbb{R}$.

Lemma 3.4. *Consider a trapping region $\mathcal{S}$. Given $\alpha \in \mathbb{R}$, $\mathcal{S}^\alpha$ is also a trapping region, that is $\mathcal{S}^\alpha$ is compact and $\mathcal{S}^{\alpha+t} \subset \operatorname{int}(\mathcal{S}^\alpha)$ for all $t > 0$.*

Proof. The compactness of $\mathcal{S}^\alpha$ follows from the compactness of $\mathcal{S}$. For the other result, by definition we have that for every $t > 0$,

$$\varphi_t(\mathcal{S}) \subset \operatorname{int}(\mathcal{S}) \Rightarrow \varphi_\alpha(\varphi_t(\mathcal{S})) \subset \varphi_\alpha(\operatorname{int}(\mathcal{S})) \Rightarrow \varphi_{\alpha+t}(\mathcal{S}) \subset \operatorname{int}(\varphi_\alpha(\mathcal{S})) \Rightarrow \mathcal{S}^{\alpha+t} \subset \operatorname{int}(\mathcal{S}^\alpha),$$

which completes the proof. □

Next, we show that q belongs to the boundary of some trapping region. The result is restricted to attracting sets for now and later extended to repelling sets.

Lemma 3.5. *Consider an attracting set $\mathcal{A} \in \mathscr{A}$ and a point $q \in \mathcal{U}_\mathcal{A}$. There exist a trapping region $\mathcal{S} \subset \mathcal{M}$ and constant $\alpha \in \mathbb{R}$, for which $q \in \operatorname{bd}(\mathcal{S}^\alpha)$.*

Proof. Consider a trapping region $\mathcal{S}$ of $\mathcal{A}$. Lemma 3.4 implies that the sequence of $\mathcal{S}^{t_i}$'s is nested for any increasing sequence of $t_i \in \mathbb{R}$, i.e., $\mathcal{S}^{t_{i+1}} \subset \mathcal{S}^{t_i}$ if $t_{i+1} > t_i$. Moreover, $\bigcap_{t>0} \mathcal{S}^t = \mathcal{A}$ and $\bigcup_{t<0} \mathcal{S}^t = \Omega_\mathcal{A}$. Hence, $\operatorname{bd}(\mathcal{S}^t) \to \operatorname{bd}(\mathcal{A})$ as $t \to \infty$ and $\operatorname{bd}(\mathcal{S}^t) \to \operatorname{bd}(\Omega_\mathcal{A})$ as $t \to -\infty$. This results in the proof since $\mathcal{S}^t$ evolves continuously in time. □

Next, consider some trapping region $\mathcal{S}$ of $\mathcal{A}$. Let $a \in \mathbb{R}$ be such that $q \in \operatorname{bd}(\mathcal{S}^a)$, which exists in view of Lemma 3.5, and consider some constant $b \in (0, |a|)$. Then $\mathcal{B}_0(q) = \mathcal{S}^{a-b} - \mathcal{S}^{a+b}$ is a neighborhood of q in $\mathcal{M}$, due to the following result.

Lemma 3.6. *Given $\alpha \in \mathbb{R}$ and $\beta > 0$, it holds that $d\big(\operatorname{bd}(\mathcal{S}^\alpha), \operatorname{bd}(\mathcal{S}^{\alpha+\beta})\big) > 0$.*

Proof. In view of Lemma 3.4, $\mathcal{S}^{\alpha+\beta} \subset \operatorname{int}(\mathcal{S}^{\alpha+\frac{\beta}{2}}) \subset \mathcal{S}^{\alpha+\frac{\beta}{2}} \subset \operatorname{int}(\mathcal{S}^\alpha)$. On the other hand, $\mathcal{S}^{\alpha+\frac{\beta}{2}}$ is a compact set. Hence, $\operatorname{bd}(\mathcal{S}^{\alpha+\frac{\beta}{2}}) \subset \operatorname{int}(\mathcal{S}^\alpha - \mathcal{S}^{\alpha+\beta})$, completing the proof. □

Finally, we construct $\mathcal{B}_\delta(q)$ based on $\mathcal{B}_0(q)$ as follows. Since $\mathcal{M}$ is a manifold, there exists some constant $\delta_\mathcal{M} > 0$ such that all points in $\mathbb{R}^n$ whose distance to $\mathcal{M}$ is not greater than $\delta_\mathcal{M}$ have a well-defined and unique projection onto $\mathcal{M}$. Given $\delta \in (0, \delta_\mathcal{M})$, define $\mathcal{B}_\delta(q)$ by

$$\mathcal{B}_\delta(q) = \mathcal{S}_\delta^{a-b} - \mathcal{S}_\delta^{a+b}$$

where for any $\sigma \in \mathbb{R}$, $\mathcal{S}_\delta^\sigma$ is defined by

$$\mathcal{S}_\delta^\sigma = \left\{ z \in \mathbb{R}^n \,\middle|\, d(z, \mathcal{S}^\sigma) < \delta,\ z_\mathcal{M} \in \mathcal{S}^\sigma \right\}$$

where $z_\mathcal{M}$ is the projection of $z \in \mathbb{R}^n$ on $\mathcal{M}$.

Lemma 3.7. *For all $\delta \in (0, \delta_\mathcal{M})$, $\mathcal{B}_\delta(q)$ is a neighborhood of q.*

Proof. Consider an n-dimensional ball centered at q with the radius

$$r = \frac{1}{2} \min\left(\delta, d\left(\operatorname{bd}(\mathcal{S}^a), \operatorname{bd}(\mathcal{S}^{a+b})\right), d\left(\operatorname{bd}(\mathcal{S}^a), \operatorname{bd}(\mathcal{S}^{a-b})\right)\right).$$

The ball is nonempty according to Lemma 3.6. On the other hand, it is a subset of the interior of $\mathcal{B}_\delta(q)$, which leads to the proof. □

Now we show the existence of a sufficiently large time, after which the trajectory through p in Theorem 3.5 never intersects $\mathcal{B}_\delta(q), \delta \in (0, \delta_\mathcal{M})$. Consider a bounded set $\mathcal{U} \subset \mathbb{R}^n$. Given a point $u \in \mathrm{bd}(\mathcal{U})$, we say *the vector field at u is pointing into $\mathcal{U}$* if there exists $\epsilon > 0$ such that the closed line segment with the end points u and $u + \epsilon f(u)$ is a subset of the closure of $\mathcal{U}$.

Proposition 3.3. *Consider an attracting set $\mathcal{A} \in \mathscr{A}$, and let q be any point in $\mathcal{U}_\mathcal{A}$. There exist $\delta_q \in (0, \delta_\mathcal{M})$ and $t_q > 0$ such that*

$$\varphi_t(p) \notin \mathcal{B}_\delta(q) \qquad \forall \delta \in (0, \delta_q), \forall t > t_q.$$

Proof. The idea of the proof is to show the existence of $\delta_q \in (0, \delta_\mathcal{M})$ and $M > 0$ such that if $\varphi_t(p) \in \mathcal{B}_\delta(q)$ for some $t > M$ and $\delta \in (0, \delta_q)$, then in finite time, $\varphi_t(p)$ enters $\mathcal{S}_r^{a+2b}$ for some $r > \delta_\mathcal{M}$, where $\mathcal{B}_\delta(q) \cap \mathcal{S}_r^{a+2b} = \emptyset$, and never returns to $\mathcal{B}_\delta(q)$. We prove in the following three steps.

Step 1. We show the existence of some $\delta^1 > 0$ such that the flow through every point $z \in \mathbb{R}^n$ whose distance to $\mathcal{B}_0(q)$ is less than δ^1, will be in $\mathcal{S}_r^{a+2b}$ for some $r > \delta_\mathcal{M}$, within a particular time interval. Consider a point $x \in \mathcal{B}_0(q)$. We first show that x reaches $\mathrm{bd}(\mathcal{S}^{a+3b})$ in finite time. According to Lemma 3.5, there exists some constant $c(x) \in \mathbb{R}$ such that $x \in \mathrm{bd}(\mathcal{S}^{c(x)})$. Then

$$\varphi_{T(x)}(x) \in \mathrm{bd}(\mathcal{S}^{a+3b}) \qquad \text{and} \qquad \varphi_t(x) \in \mathcal{S}^{a+3b} \quad \forall t \geq T(x) \tag{3.6}$$

where $T(x) = a + 3b - c(x)$ and is finite since $a - b \leq c(x) \leq a + b$. Next, from Theorem 3.2, there exists $\delta_x^1 > 0$ such that for all $t \in [\sup_{x \in \mathcal{B}_0(q)} T(x), \sup_{x \in \mathcal{B}_0(q)} T(x) + b]$,

$$\|z - x\| < \delta_x^1 \Rightarrow \|\varphi_t(z) - \varphi_t(x)\| < \underbrace{\frac{1}{2} d\left(\mathrm{bd}(\mathcal{S}^{a+2b}), \mathrm{bd}(\mathcal{S}^{a+3b})\right)}_{r}. \tag{3.7}$$

The constant r is positive since $d\left(\mathrm{bd}(\mathcal{S}^{a+2b}), \mathrm{bd}(\mathcal{S}^{a+3b})\right) > 0$ in view of Lemma 3.6. Now based on the second statement in (3.6), from (3.7) we obtain that for all $t \in [\sup_{x \in \mathcal{B}_0(q)} T(x), \sup_{x \in \mathcal{B}_0(q)} T(x) + b]$,

$$\|z - x\| < \delta_x^1 \Rightarrow \varphi_t(z) \in \mathrm{int}(\mathcal{S}_r^{a+2b}). \tag{3.8}$$

On the other hand, since $\mathcal{B}_0(q)$ does not include any fixed point and the vector field is continuous, $\delta^1 = \min_{x \in \mathcal{B}_0(q)} \delta_x^1$ is well defined and positive. Then (3.8) yields that for all $t \in [\sup_{x \in \mathcal{B}_0(q)} T(x), \sup_{x \in \mathcal{B}_0(q)} T(x) + b] = [4b, 5b]$,

$$d(z, \mathcal{B}_0(q)) < \delta^1 \Rightarrow \varphi_t(z) \in \mathrm{int}(\mathcal{S}_r^{a+2b}). \tag{3.9}$$

Step 2. We show the existence of some $\delta^2 \in (0, \delta_\mathcal{M})$ such that the projection onto $\mathcal{M}$ of the vector field at every point in the side-boundary $\mathcal{D}_{\delta^2}$ of $\mathcal{S}_{\delta^2}^{a+b}$, where $\mathcal{D}_y = \{z \in \mathbb{R}^n \,|\, d(z, \mathcal{S}^{a+b}) < y,\, z_\mathcal{M} \in \mathrm{bd}(\mathcal{S}^{a+b})\}$ for $y > 0$, is pointing into $\mathcal{S}_{\delta^2}^{a+b}$. Consider some point $x \in \mathrm{bd}(\mathcal{S}^{a+b})$. In view of Lemma 3.4, the vector field at x is pointing into $\mathcal{S}^{a+b}$. Hence, due to the continuity of the vector field, there exists $\delta_x^2 \in (0, \delta_\mathcal{M})$ such that the projection of the vector field at every point in $\mathcal{D}_{\delta_x^2}$ onto $\mathcal{M}$ is pointing into $\mathcal{S}_{\delta_x^2}^{a+b}$. Since $\mathrm{bd}(\mathcal{S}^{a+b})$ does not include any fixed point and the vector field is continuous, $\delta^2 = \min_{x \in \mathrm{bd}(\mathcal{S}^{a+b})} \delta_x^2$ is well-defined and positive. Hence, $\mathcal{S}_{\delta^2}^{a+b}$ is well-defined and

the projection onto $\mathcal{M}$ of the vector field at every point on its side-boundary $\mathcal{D}_{\delta^2}$ is pointing into $\mathcal{S}_{\delta^2}^{a+b}$.

Step 3. Let $\delta_q = \min(\delta^1, \delta^2)$, and consider some $\delta \in (0, \delta_q)$. The fact that $\varphi_t(p) \to \mathcal{M}$ as $t \to \infty$ implies the existence of $M > 0$ such that

$$d\left(\varphi_t(p), \mathcal{M}\right) < \delta \qquad \forall t > M. \tag{3.10}$$

If $\varphi_t(p) \notin \mathcal{B}_\delta(q)$ for all $t > M$, the result is trivial by taking $t_q = M$. So consider the case when there exists some $t^1 > M$ such that $\varphi_{t^1}(p) \in \mathcal{B}_\delta(q)$. Then (3.9) yields the existence of $t^2 > t^1$ such that

$$\varphi_{t^2}(p) \in \operatorname{int}(\mathcal{S}_r^{a+2b}). \tag{3.11}$$

Now consider some time $t^3 > t^2$. Towards contradiction, we show $\varphi_{t^3}(p) \notin \mathcal{B}_\delta(q)$. Assume on the contrary that $\varphi_{t^3}(p) \in \mathcal{B}_\delta(q)$. Then from (3.11) and since $a + b < a + 2b$ and $\mathcal{B}_\delta(q)$ is open on its side-boundary $\mathcal{D}_\delta$, it holds that $\varphi_t(p)_{\mathcal{M}}$ must pass $\operatorname{bd}(\mathcal{S}^{a+b})$ from inside at some time $t^4 \in (t^2, t^3)$ due to the Jordan–Brouwer separation theorem [6]. Consequently, $\varphi_t(p)$ must pass $\mathcal{D}_\delta$ at t^4 from inside and exit $\mathcal{S}_\delta^{a+b}$. This is a contradiction since the projection onto $\mathcal{M}$ of the vector field is pointing into $\mathcal{S}_\delta^{a+b}$ at every point on its side-boundary $\mathcal{D}_\delta$. Therefore, $\varphi_t(p) \notin \mathcal{B}_\delta(q)$ for all $t > t^2$, which leads to the proof by taking $t_q = t^2$. □

Next, if $\mathscr{R} \neq \emptyset$, we focus on an arbitrary point $q \in \mathcal{U}_\mathcal{R}$ for some repelling set $\mathcal{R} \in \mathscr{R}$, and define $\mathcal{C}_\delta(q)$ similar to $\mathcal{B}_\delta(q)$. Correspondingly, we can state the following result, the proof of which is similar to that of Proposition 3.3.

Proposition 3.4. *Consider a repelling set $\mathcal{R} \in \mathscr{R}$, and let q be any point in $\mathcal{U}_\mathcal{R}$. There exist $\delta_q \in (0, \delta_\mathcal{M})$ and $t_q > 0$ such that*

$$\varphi_t(p) \notin \mathcal{C}_\delta(q) \qquad \forall \delta \in (0, \delta_q), \forall t > t_q.$$

Now we focus on the semi-attracting sets; the case with semi-repelling sets can be handled similarly. We follow the same steps as those for attracting sets to build a neighborhood of an arbitrary point q in $\mathcal{U}_{\mathcal{A}'}$ for some $\mathcal{A}' \in \mathscr{A}'$.

Lemma 3.8. *Consider a semi-trapping region $\mathcal{S}$. Given $\alpha \in \mathbb{R}$, $\mathcal{S}^\alpha$ is also a semi-trapping region, that is it is compact, $\mathcal{S}^\alpha \cap \operatorname{bd}(\mathcal{M}) \neq \emptyset$, and the set $\hat{\mathcal{S}}^\alpha = \mathcal{S}^\alpha \cap \operatorname{int}(\mathcal{M})$ satisfies $\varphi_t(\hat{\mathcal{S}}^\alpha) \subset \operatorname{int}(\hat{\mathcal{S}}^\alpha)$ for all $t > 0$.*

Proof. According to the definition of a semi-trapping region we have that

$$\mathcal{S} \cap \operatorname{bd}(\mathcal{M}) \neq \emptyset \Rightarrow \varphi_\alpha(\mathcal{S} \cap \operatorname{bd}(\mathcal{M})) \neq \emptyset \Rightarrow \varphi_\alpha(\mathcal{S}) \cap \varphi_\alpha(\operatorname{bd}(\mathcal{M})) \neq \emptyset. \tag{3.12}$$

Now since $\mathcal{M}$ is invariant, its boundary is so as well. Therefore, (3.12) implies

$$\varphi_\alpha(\mathcal{S}) \cap \operatorname{bd}(\mathcal{M}) \neq \emptyset \Rightarrow \mathcal{S}^\alpha \cap \operatorname{bd}(\mathcal{M}) \neq \emptyset.$$

The rest of the proof is similar to that of Lemma 3.4. □

Then similar to Lemma 3.5, we have the following result.

Lemma 3.9. *Consider a semi-attracting set $\mathcal{A}' \in \mathscr{A}'$ and a point $q \in \mathcal{U}_{\mathcal{A}'}$. There exist a semi-trapping region $\mathcal{S} \subset \mathcal{M}$ and a constant $\alpha \in \mathbb{R}$, for which $q \in \mathrm{bd}(\mathcal{S}^\alpha)$.*

Next, consider some semi-trapping region $\mathcal{S}$ of $\mathcal{A}'$ and let $a \in \mathbb{R}$ be such that $q \in \mathrm{bd}(\mathcal{S}^a)$, which exists in view of Lemma 3.9, and consider some constant $b \in (0, |a|)$. Then it can be shown that again $\mathcal{B}_0(q) = \mathcal{S}^{a-b} - \mathcal{S}^{a+b}$ is a neighborhood of q in $\mathcal{M}$. Correspondingly, for a given $\delta \in (0, \delta_{\mathcal{M}})$, we construct the neighborhood $\mathcal{B}'_\delta(q)$ of q in $\mathbb{R}^n$ as

$$\mathcal{B}'_\delta(q) = \left(\mathcal{S}_\delta^{a-b} - \mathcal{S}_\delta^{a+b}\right) \cup \left(\tilde{\mathcal{S}}_\delta^{a-b} - \tilde{\mathcal{S}}_\delta^{a+b}\right)$$

where for $\sigma \in \mathbb{R}$,

$$\tilde{\mathcal{S}}_\delta^\sigma = \left\{ z \in \mathbb{R}^n \,\middle|\, d(z, (\mathrm{bd}(\mathcal{M}) \cap \mathcal{S}^\sigma)) < \delta \right\}.$$

Similar to Lemma 3.7, we can state the following based on the construction of $\mathcal{B}'_\delta(q)$.

Lemma 3.10. *For all $\delta \in (0, \delta_{\mathcal{M}})$, $\mathcal{B}'_\delta(q)$ is a neighborhood of q.*

Now we proceed to the main result for semi-attracting sets.

Proposition 3.5. *Consider a semi-attracting set $\mathcal{A}' \in \mathscr{A}'$, and let q be any point in $\mathcal{U}_{\mathcal{A}'}$. There exist $\delta_q \in (0, \delta_{\mathcal{M}})$ and $t_q > 0$ such that*

$$\varphi_t(p) \notin \mathcal{B}'_\delta(q) \qquad \forall \delta \in (0, \delta_q), \forall t > t_q.$$

Proof. The proof follows the same steps as those in that of Proposition 3.3, and we use some of the notations here as well.

Step 1. One can show the existence of $\delta_x^1 > 0$ such that the following holds for all $t \in [\sup_{x \in \mathcal{B}_0(q)} T(x), \sup_{x \in \mathcal{B}_0(q)} T(x) + b] = [4b, 5b]$:

$$\|z - x\| < \delta_x^1 \Rightarrow \varphi_t(z) \in \mathcal{S}_r^{a+2b} \cup \tilde{\mathcal{S}}_{\delta^3}^{a+2b}$$

where $r = \frac{1}{2} d\left(\mathrm{bd}(\mathcal{S}^{a+2b}), \mathrm{bd}(\mathcal{S}^{a+3b})\right)$ and $\delta^3 = \frac{1}{2} d\left(\mathrm{bd}(\mathcal{S}^{a+2b}), \mathrm{bd}(\mathcal{S}^{a+b})\right)$. Then by defining $\delta^1 = \min_{x \in \mathcal{B}_0(q)} \delta_x^1$, it holds that for all $t \in [4b, 5b]$,

$$d(z, \mathcal{B}_0(q)) < \delta^1 \Rightarrow \varphi_t(z) \in \mathcal{S}_r^{a+2b} \cup \tilde{\mathcal{S}}_{\delta^3}^{a+2b}. \tag{3.13}$$

Step 2. One can show the existence of some $\delta^2 \in (0, \delta_{\mathcal{M}})$ such that the projection onto $\mathcal{M}$ of the vector field at every point on the boundary $\mathcal{D}_{\delta^2}$ of $\mathcal{S}_{\delta^2}^{a+b}$ is pointing into $\mathcal{S}_{\delta^2}^{a+b}$.

Step 3. Let $\delta_q = \min(\delta^1, \delta^2, \delta^3)$, and consider some $\delta \in (0, \delta_q)$. The fact that $\varphi_t(p) \to \mathcal{M}$ as $t \to \infty$ implies the existence of $M > 0$ such that

$$d\left(\varphi_t(p), \mathcal{M}\right) < \delta \qquad \forall t > M. \tag{3.14}$$

If $\varphi_t(p) \notin \mathcal{B}'_\delta(q)$ for all $t > M$, the result is trivial by taking $t_q = M$. So consider the case when there exists some $t^1 > M$ such that $\varphi_{t^1}(p) \in \mathcal{B}'_\delta(q)$. Then (3.15) yields the existence of $t^2 > t^1$ such that

$$\varphi_{t^2}(p) \in \mathcal{S}_r^{a+2b} \cup \tilde{\mathcal{S}}_{\delta^3}^{a+2b}. \tag{3.15}$$

Now consider some time $t^3 > t^2$. Towards contradiction, we show $\varphi_{t^3}(p) \notin \mathcal{B}'_\delta(q)$. Assume on the contrary that $\varphi_{t^3}(p) \in \mathcal{B}'_\delta(q)$. Then from (3.15) and since $a+b < a+2b$ and $\mathcal{B}'_\delta(q)$ is open on its boundary $\mathcal{D}_\delta$, $\varphi_t(p)_{\mathcal{M}}$ must pass either $\mathrm{bd}(\mathcal{S}^{a+b})$ from inside or $\mathrm{bd}(\tilde{\mathcal{S}}^{a+b})$ at some time $t^4 \in (t^2, t^3)$ due to the Jordan–Brouwer separation theorem [6]. The second case is a contradiction since $\delta < \delta_3$, which in view of (3.14) implies that $\varphi_t(p)$ can never pass from $\mathrm{bd}(\tilde{\mathcal{S}}^{a+b})$ and enter $\mathcal{B}'_\delta(q)$ for $t > M$. On the other hand, the first case is a contradiction since the projection onto $\mathcal{M}$ of the vector field is pointing into $\mathcal{S}^{a+b}_\delta$ at every point on its boundary $\mathcal{D}_\delta$. Therefore, $\varphi_t(p) \notin \mathcal{B}'_\delta(q)$ for all $t > t^2$, which leads to the proof by taking $t_q = t^2$. □

Now we proceed to the proof of Theorem 3.5.

Proof of Theorem 3.5. First, we prove by contradiction that

$$\omega(p) \cap \Big(\bigcup_{\mathcal{A}\in\mathscr{A}} \mathcal{U}_{\mathcal{A}} \bigcup_{\mathcal{R}\in\mathscr{R}} \mathcal{U}_{\mathcal{R}} \bigcup_{\mathcal{A}'\in\mathscr{A}'} \mathcal{U}_{\mathcal{A}'} \bigcup_{\mathcal{R}'\in\mathscr{R}'} \mathcal{U}_{\mathcal{R}'} \Big) = \emptyset. \tag{3.16}$$

Assume on the contrary that there exists some point q such that $q \in \omega(p)$ and either $q \in \mathcal{U}_{\mathcal{A}}$ for some $\mathcal{A} \in \mathscr{A}$, $q \in \mathcal{U}_{\mathcal{R}}$ for some $\mathcal{R} \in \mathscr{R}$, $q \in \mathcal{U}_{\mathcal{A}'}$ for some $\mathcal{A}' \in \mathscr{A}'$ or $q \in \mathcal{U}_{\mathcal{R}'}$ for some $\mathcal{R}' \in \mathscr{R}'$. The first case implies that a point in $\mathcal{U}_{\mathcal{A}}$ is an ω limit point of p, which is impossible in view of Proposition 3.3 and Lemma 3.7. Similarly, the other cases result in a contradiction. Hence, (3.16) holds. So $\omega(p) \subseteq \mathcal{N}$. On the other hand, $\omega(p)$ is bounded since $\varphi_t(p) \to \mathcal{M}$ as $t \to \infty$. Hence, in view of Lemma 1.1, $\omega(p)$ is connected. So there exists a connected component $\mathcal{N}_i$ of $\mathcal{N}$ such that $\omega(p) \subseteq \mathcal{N}_i$, completing the proof. □

One interesting possible application of Theorem 3.5 is to repeatedly apply it to a single manifold. For this, given an invariant compact manifold $\mathcal{M}$, we first characterize $\mathcal{N}$ containing all candidate limit sets. Then Theorem 3.5 implies convergence to $\mathcal{N}$ that in view of Proposition 3.2 is itself an invariant compact manifold. Hence, Theorem 3.5 can again be applied, but this time to $\mathcal{N}$ instead of $\mathcal{M}$. Namely, we find the (semi-)attracting and (semi-)repelling sets of $\mathcal{N}$ and subtract their corresponding $\mathcal{U}_{\mathcal{A}}, \mathcal{U}_{\mathcal{R}}, \mathcal{U}_{\mathcal{A}'}$ and $\mathcal{U}_{\mathcal{R}'}$ from $\mathcal{N}$ to get a new $\mathcal{N}$, say $\mathcal{N}'$, to which $\varphi(p)$ converges. This process can be repeated until no further reduction of $\mathcal{N}$ is possible. As an illustration, by applying Theorem 3.5 for the second time to the set $\mathcal{N}$ in Fig. 3.4, we obtain $\mathcal{N}'$ in Fig. 3.5 where the converging trajectory may converge to.

Following this approach, we obtain, for example, the following slightly-weaker version of Corollary 3.1 in Section 3.3, indicating the strength of Theorem 3.5.

Corollary 3.4. *Consider the vector field* (3.1)*, and let p be an arbitrary point in $\mathbb{R}^n$. If $\varphi_t(p)$ converges to a one-dimensional invariant compact manifold $\mathcal{M}$, in the form of a compact, simple open curve, then $\omega(p)$ is a single fixed point or a continuum of fixed points on $\mathcal{M}$.*

Proof. Consider one of the ending points z of $\mathcal{M}$, which is a fixed point since $\mathcal{M}$ is invariant. The fixed point z is either a singleton or belongs to a continuum of fixed points in $\mathcal{M}$. In the first case, we define $\mathcal{Y} = \{z\}$, and in the second case we define $\mathcal{Y}$ as the whole continuum. If $\mathcal{Y} = \mathcal{M}$, the result is trivial; otherwise, $\mathcal{Y}$ is either a semi-attracting or a semi-repelling set of f in $\mathcal{M}$. Therefore, the orbit connected to $\mathcal{Y}$, say $\mathcal{O}^{\mathcal{Y}}$, does not belong to $\mathcal{N}$. Hence, in view of Theorem 3.5, $\varphi_t(p)$ converges to $\mathcal{M} - \mathcal{O}^{\mathcal{Y}}$ which is an invariant compact manifold. So by repeating the

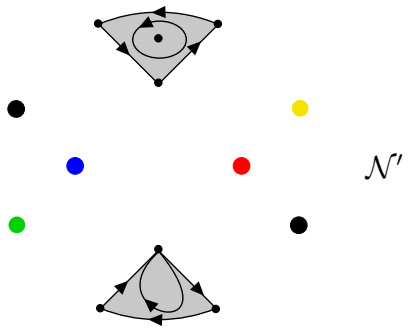


Figure 3.5: The set $\mathcal{N}'$ after applying Theorem 3.5 for the second time to the original set $\mathcal{N}$ in Fig. 3.4. The two big black fixed points were semi-attracting and semi-repelling sets of f in (the original) $\mathcal{N}$, respectively. Hence, the two non-fixed orbits connected to them have no intersection with $\omega(p)$ and can be ruled out from $\mathcal{N}$, resulting in $\mathcal{N}'$. Repeating this procedure will not further reduce $\mathcal{N}'$.

above procedure for $\mathcal{M} - \mathcal{O}^{\mathcal{Y}}$, we rule out another non-fixed orbit connected to the ending single fixed point or continuum of fixed points in $\mathcal{M} - \mathcal{O}^{\mathcal{Y}}$. Therefore, the repetition of this process will eventually rule out all non-fixed orbits from $\omega(p)$, completing the proof. □

3.6 When $\mathcal{M}$ is diffeomorphic to a plane and $f|_{\mathcal{M}}$ has only hyperbolic singularities

Although the boundary of each $\mathcal{N}_i$ consists of orbits, its topology may be as complicated as, for example, the union of infinitely many regions, each surrounded by a homoclinic orbit of the same fixed point. However, the vector field does not exhibit such complicated phenomenons if $f|_{\mathcal{M}}$ is hyperbolic. Given an invariant manifold $\mathcal{M} \subset \mathbb{R}^n$, we call a fixed point $p \in \mathcal{M}$ *hyperbolic under* $f|_{\mathcal{M}}$, if every eigenvalue of the Jacobian matrix of $f|_{\mathcal{M}}$ at p, has a non-zero real part. This is a weaker notion of hyperbolicity in general: if $p \in \mathcal{M}$ is a hyperbolic fixed point of f, then it is also hyperbolic under $f|_{\mathcal{M}}$, but not necessarily the other way around. When in addition to the hyperpolicity of $f|_{\mathcal{M}}$, the compact invariant manifold $\mathcal{M}$ is two-dimensional, the possibilities for the limit set of a trajectory converging to $\mathcal{M}$ will be limited to the boundaries of the (semi-)attracting and (semi-)repelling sets or those of their attraction and repulsion regions, as stated in the following. Moreover, intuitively each closed subcurve of the limit set must be compactly oriented, so that if we start from one point on the subcurve, we return to it after passing through the subcurve. After making the hyperbolicity assumption, this leads to the following definition that is a special case of flow-aligned curves for general $\mathbf{C}^1$ vector fields in Section 3.3.

Definition 3.7 (hyperbolic flow-aligned curves)**.** *Under the vector field* (3.1)*, consider an invariant, compact, simple closed curve* Γ*. We say* Γ *is a* hyperbolic flow-aligned simple curve*, if i) every fixed point on* Γ *is hyperbolic under* $f|_\Gamma$ *and ii)* Γ *is a closed orbit or each fixed point on* Γ *is the* ω *limit set of exactly one orbit in* Γ *and the* α *limit set of exactly one (not necessarily different) orbit in* Γ*. A connected union of hyperbolic flow-aligned simple curves is called a* hyperbolic flow-aligned curve.

The second case in the definition corresponds to a number of fixed points and compactly oriented connecting orbits. Now we present the main result of this section.

Theorem 3.6. *Consider the vector field* (3.1) *possessing a two-dimensional invariant compact manifold $\mathcal{M} \subset \mathbb{R}^n$ diffeomorphic to a plane. Suppose that every fixed point on $\mathcal{M}$ is hyperbolic under $f|_{\mathcal{M}}$. Let p be an arbitrary point in $\mathbb{R}^n$. If $\varphi_t(p) \to \mathcal{M}$ as $t \to \infty$, then $\omega(p)$ is either a single fixed point or a hyperbolic flow-aligned curve formed by the boundaries of a number of*

- *attracting, repelling, semi-attracting and semi-repelling sets in $\mathcal{M}$, and*
- *attracting, repelling, semi-attracting and semi-repelling set's attraction region in $\mathcal{M}$.*

We start the proof by determining the possible structure of the candidate limit sets $\mathcal{N}_i$. The hyperbolicity assumption on $\mathcal{M}$ leaves no interior for any $\mathcal{N}_i$. We show this in Lemma 3.11, but for that we need the following result.

Proposition 3.6. *Consider a $\mathbf{C}^\ell, \ell \geq 1$, planar vector field, and let $\mathcal{M}$ be a positively invariant compact set in $\mathbb{R}^2$, containing finitely many fixed points. Let Γ denote a connected union of flow-aligned closed curves γ_i and $\mathcal{G}$ be the union of the simply-connected compact sets whose boundaries are γ_i. If for some point $p \in \mathcal{M} - \mathcal{G}$ we have that $\omega(p) = \Gamma$, then $\mathcal{G}$ is an attracting set.*

Proof. First we prove that $\mathcal{G}$ is stable. For $\delta > 0$, define the neighborhood $\mathcal{Y}_\Gamma^\delta$ of Γ by

$$\mathcal{Y}_\Gamma^\delta = \left\{ z \in \mathbb{R}^2 \,|\, d(z, \Gamma) < \delta \right\}.$$

Now consider an arbitrary neighborhood $\mathcal{U}$ of $\mathcal{G}$. There exists $\delta_1 > 0$ such that $\mathcal{Y}_\Gamma^{\delta_1} \subset \mathcal{U}$. Moreover, since $\mathcal{M}$ contains finitely many fixed points, there exists $\delta_2 > 0$ such that $\mathcal{Y}_\Gamma^{\delta_2}$ contains no other closed orbit or fixed point but those in $\mathcal{G}$. Let $\delta_3 = \min\{\delta_1, \delta_2\}$. The fact that $\omega(p) = \Gamma$, implies $\varphi_t(p) \to \Gamma$ as $t \to \infty$. Hence, there exists some $M > 0$ such that

$$\phi_t(p) \in \mathcal{Y}_\Gamma^{\delta_3} \qquad \forall t > M. \tag{3.17}$$

Again since $\mathcal{M}$ contains finitely many fixed points, there exists a non-fixed point q on Γ, and one can construct a transverse segment $\mathcal{L}$ through it. From Lemma 2.2, $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ infinitely many times. Since the sequence of intersections converges to q, there exists some intersection q_1 after $t = M$ such that the closed line segment $q_1 q$ is a subset of $\mathcal{Y}_\Gamma^{\delta_3}$. Denote by q_2 the first intersection after q_1. The intersections q_1 and q_2 are distinct since otherwise, $\omega(p)$ is a closed orbit, which is in contradiction with $\omega(p) = \Gamma \subset \mathcal{G} \not\ni p$. Denote that part of $\mathcal{O}_+(p)$ confined to q_1 and q_2 by $\mathcal{O}_{q_1,q_2}(p)$. Then the simply-connected compact set $\mathcal{V}$ whose boundary is the union of $\mathcal{O}_{q_1,q_2}(p)$ and that part of $\mathcal{L}$ between q_1 and q_2, is positively invariant. Moreover, in view of (3.17), $\mathcal{O}_{q_1,q_2}(p) \subset \mathcal{Y}_\Gamma^{\delta_3}$, which together with $q_1 q \subset \mathcal{Y}_\Gamma^{\delta_3}$ imply that

$$\mathcal{V} \subset \mathcal{Y}_\Gamma^{\delta_3} \subset \mathcal{U}. \tag{3.18}$$

Now $\mathcal{V} \cap \Gamma \neq \emptyset$ since otherwise, $\mathcal{V}$ does not contain any fixed point, which is impossible since then for any point $v \in \mathcal{V}$, $\omega(v) = \emptyset$, a contradiction in view of Lemma 1.1. On the other hand, the fact that Γ equals $\omega(p)$, which is invariant according to Lemma 1.1, implies that $\mathrm{bd}(\mathcal{V})$ and Γ never

intersect. Therefore, $\Gamma \subset \mathcal{V}$, resulting in $\mathcal{G} \subset \mathcal{V}$. Hence, corresponding to every neighborhood $\mathcal{U}$ of $\mathcal{G}$, there exists a neighborhood $\mathcal{V}$ of $\mathcal{G}$ such that all trajectories that start in $\mathcal{V}$ stay in $\mathcal{V}$, and hence in $\mathcal{U}$ according to (3.18), for all $t > 0$. This proves the stability of $\mathcal{G}$.

Next, we prove the asymptotic stability of $\mathcal{G}$ by showing that every trajectory in $\mathcal{V}$ converges to $\mathcal{G}$. It suffices to show that for an arbitrary point $v \in \mathcal{V} - \mathcal{G}$, $\omega(v) = \Gamma$. Clearly $\omega(v)$ is a subset of $\mathcal{V}$ and in view of Lemma 1.1 is nonempty. Should $\omega(v)$ be a closed orbit, it would have to be Γ since there are no closed orbits in $\mathcal{Y}_\Gamma^{\delta_3} - \mathcal{G}$. So assume otherwise. Through a proof by contradiction, we show that $\omega(v)$ is not a single fixed point. Assume on the contrary that $\omega(v) = \{g\}$ for some fixed point $g \in \mathcal{V}$. From (3.18) and since the only fixed points in $\mathcal{Y}_\Gamma^{\delta_3}$ are those in $\mathcal{G}$, we obtain $g \in \mathcal{G}$. Similar to what was shown in the first part, one can show the existence of two consecutive intersections of $\mathcal{O}_+(p)$ with $\mathcal{L}$, say q_i and q_{i+1}, such that v does not belong to the simply-connected compact set $\hat{\mathcal{V}}$ whose boundary is the union of $\mathcal{O}_{q_i,q_{i+1}}(v)$ and that part of $\mathcal{L}$ between q_i and q_{i+1}. Hence, the orbit connecting v to g, denoted by ν, starts out of $\hat{\mathcal{V}}$, intersects $\mathcal{L}$ at some point and converges to g. Denote the last intersection of $O_+(v)$ with $\mathcal{L}$ by l. Then the simply-connected compact set $\mathcal{W}$, whose boundary is the union of that part of ν from l to g, that part of $\mathcal{L}$ between l and q, and that part of Γ from q to g is positively invariant. This implies that $\mathcal{O}_+(q_{i+1})$ and hence $\mathcal{O}_+(p)$ never reaches $\Gamma - \mathcal{W}$, contradicting $\omega(p) = \Gamma$. So $\omega(v)$ is not a single fixed point. Then according to Theorem 2.2, $\omega(v)$ consists of a finite number of fixed points and orbits connecting them, all in $\mathcal{V}$. However, the only fixed points that $\mathcal{O}_+(v)$ may converge to are those on Γ (not the ones inside) since Γ is invariant. So $\omega(v)$ consists of a finite number of fixed points on Γ and orbits in $\mathcal{V}$ connecting them. From contradiction, it can be shown that every non-fixed orbit in $\omega(v)$, connecting two (not necessarily the same) fixed points of Γ, also belongs to $\omega(p)$, and vice verse. Therefore, $\omega(v) = \Gamma$, and since $\mathcal{G}$ is stable, we conclude that it is asymptotically stable, implying that it is an attracting set in view of Lemma 3.1, completing the proof. □

Lemma 3.11. *Assume that the conditions in Theorem 3.6 are satisfied. Then, for every candidate limit set $\mathcal{N}_i$, it holds that* $\text{int}(\mathcal{N}_i) = \emptyset$*, or equivalently* $\mathcal{N}_i = \text{bd}(\mathcal{N}_i)$.

Proof. We prove by contradiction. Assume on the contrary that there exists an $\mathcal{N}_i$ whose interior is nonempty. First we show that the only possible fixed points in $\mathcal{N}_i$ are hyperbolic saddles. By definition, no sink or source fixed point belongs to $\mathcal{N}_i$; otherwise, $\mathcal{N}_i$ is the source or fixed point itself, which is impossible since then $\text{int}(\mathcal{N}_i) = \emptyset$. On the other hand, the hyperbolic assumption on $\mathcal{M}$ implies that every fixed point is either a hyperbolic saddle, source or sink. So the only possible fixed points in $\mathcal{N}_i$ are hyperbolic saddles.

Next, the hyperbolic assumption on $\mathcal{M}$ implies that $\mathcal{M}$ contains finitely many isolated fixed points [76]. Hence, $\mathcal{N}_i$ contains finitely many hyperbolic saddle points, each having exactly one one-dimensional invariant stable manifold. Therefore, there are finitely many invariant stable manifolds in $\mathcal{N}_i$. So there exists a point $y \in \text{int}(\mathcal{N}_i)$ that does not lie on any of the stable manifolds. On the other hand, according to Corollary 3.3, $\mathcal{N}_i$ is invariant. Hence, in view of Theorem 2.2, $\omega(y)$ is nonempty and takes one of the three forms therein. However, $\omega(y)$ does not equal a single fixed point since it does not lie on the stable manifold of any saddle point in $\mathcal{N}_i$. Moreover, $\omega(y)$ cannot be a closed orbit γ since then γ with its inside constructs an attracting (or repelling) set whose domain of attraction (or repulsion) includes y, implying that $y \notin \mathcal{N}$ and hence $y \notin \mathcal{N}_i$. Similarly, $\omega(y)$ cannot be a union of fixed points and connecting orbits γ since then in view of Proposition 3.6, again the union with its inside constructs an attracting set if y is located outside of the union.

The same can be shown if y is located inside the union. Hence, $\omega(y)$ does not take any of the three forms in Theorem 2.2, a contradiction. Therefore, $\text{int}(\mathcal{N}_i) = \emptyset$.

On the other hand, $\mathcal{N}_i$ is closed since in (3.5), $\mathcal{M}$ is closed, each of $\mathcal{U}_\mathcal{A}$ and $\mathcal{U}_\mathcal{R}$ is open, and that for each $\mathcal{U}_{\mathcal{A}'}$ and $\mathcal{U}_{\mathcal{R}'}$, both of its intersections with $\text{int}(\mathcal{M})$ and $\text{bd}(\mathcal{M})$ are open. So $\mathcal{N}_i = \text{bd}(\mathcal{N}_i)$, completing the proof. □

Next, we show that $\omega(p)$ is either a single fixed point or limited to the boundaries of the two cases mentioned in Theorem 3.6.

Lemma 3.12. *Under the conditions in Theorem 3.6, $\omega(p)$ is either a single fixed point or a connected union of nonempty subsets of the boundaries of the eight sets mentioned in Theorem 3.6.*

Proof. The result is trivial if $\omega(p)$ is a single fixed point, so consider otherwise. According to Theorem 3.5, there exists an $\mathcal{N}_i$ such that $\omega(p) \subseteq \mathcal{N}_i$. Moreover, $\mathcal{N}_i = \text{bd}(\mathcal{N}_i)$, due to Lemma 3.11. Now according to the definition of $\mathcal{N}_i$, $\text{bd}(\mathcal{N}_i)$ is a subset of $\text{bd}(\mathcal{M})$ and the union of the boundaries of the eight types of sets stated in Theorem 3.6. However, $\text{bd}(\mathcal{M})$ is itself a subset of the boundaries of the eight types; otherwise, it contains part of the boundary of a set that is neither a semi-attracting nor a semi-repelling set or their attraction and repulsion regions, which by definition of $\mathcal{N}$, implies the existence of an $\mathcal{N}_j$ whose interior is nonempty, which is impossible. So $\omega(p)$ is a nonempty subset of the union of the boundaries of the two cases mentioned in Theorem 3.6. This leads to the proof since $\omega(p)$ is connected according to Lemma 1.1. □

Finally, we prove Theorem 3.6 by invoking Theorem 3.4 in Section 3.3.

Proof of Theorem 3.6. In view of Lemma 3.12, we only need to consider the case when $\omega(p)$ equals the union of nonempty subsets of the boundaries of the eight sets mentioned in Theorem 3.6, and show that the union is indeed a hyperbolic flow-aligned closed curve. The hyperbolicity of the union is obvious, so we only need to show the flow-alignedness. Since each of the eight sets mentioned for $\omega(p)$ in Theorem 3.6 is a bounded, simple invariant curve, $\varphi_t(p)$ converges to a bounded invariant curve Γ. On the other hand, no continuum of fixed points shows up in Γ due to the hyperbolicity assumption on $\mathcal{M}$. Therefore, from Theorem 3.4 in Section 3.3, $\omega(p)$ is either a single fixed point or the union of simple (hyperbolic) flow-aligned sub-curves of Γ, leading to the proof. □

3.7 Examples

Example 3.3. We show the limit set of a trajectory approaching an invariant circle. Let (r, θ, z), $r \geq 0$, be cylindrical coordinates, and consider the $\mathbf{C}^1$ vector field

$$\begin{aligned} \dot{r} &= -\frac{1}{2}z^2(r-1) \\ \dot{\theta} &= 10z \\ \dot{z} &= -\frac{1}{2}z^3 \end{aligned} \tag{3.19}$$

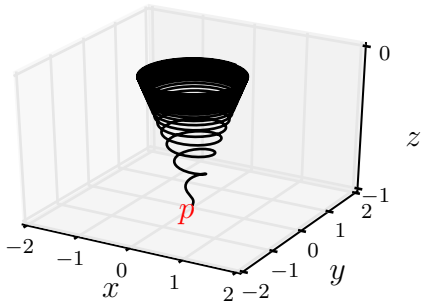


Figure 3.6: Phase portrait, in Cartesian coordinates, of (3.19) with the initial condition $(r, \theta, z) = (0, 0, -1)$. The plane $\{z = 0\}$ is full of fixed points and the shown trajectory converges to it. However, from the dynamics the unit circle centered at the origin is invariant and forms the associated limit set.

The orbit through $p = (r, \theta, z) = (0, 0, -1)$ converges to the unit circle in the invariant plane $\{z = 0\}$, centered at the origin. In view of Section 3.5, this unit circle can be taken as the invariant compact set $\mathcal{M}$. It then follows that $\mathcal{N} = \mathcal{M}$ since there are no (semi-)attractors or (semi-)repellors in $\mathcal{M}$. Therefore, according to Theorem 3.5, $\varphi_t(p)$ converges to the whole $\mathcal{N}$ which is the unit circle. The corresponding phase-portrait is shown in Fig. 3.6.

3.7.1 Application to the replicator dynamics

We illustrate the developed results by the well-known population dynamics, the replicator dynamics (1.2), mainly used to explain the evolution of species in biology [149, 68, 179, 135]. As in other population dynamics, it may often happen that a species vanishes in the long run under the replicator dynamics, i.e., $x_i(t) \to 0$ as $t \to \infty$ for some $i = 1, \dots, n$. For example, the following result that is a modified version of [181, Proposition 3.2], implies that if there is a mixture of species, say $y \in \Delta$, that outperforms some species i against any population vector, then the population portion of species i converges to zero, provided that all species are initially present in the population.

Proposition 3.7. *[181] Consider the dynamics* (1.2)*, and let* $i \in \{1, \dots, n\}$*. If there exists* $y \in \Delta$ *such that*

$$u(y, z) > u(e^i, z) \qquad \forall z \in \Delta,$$

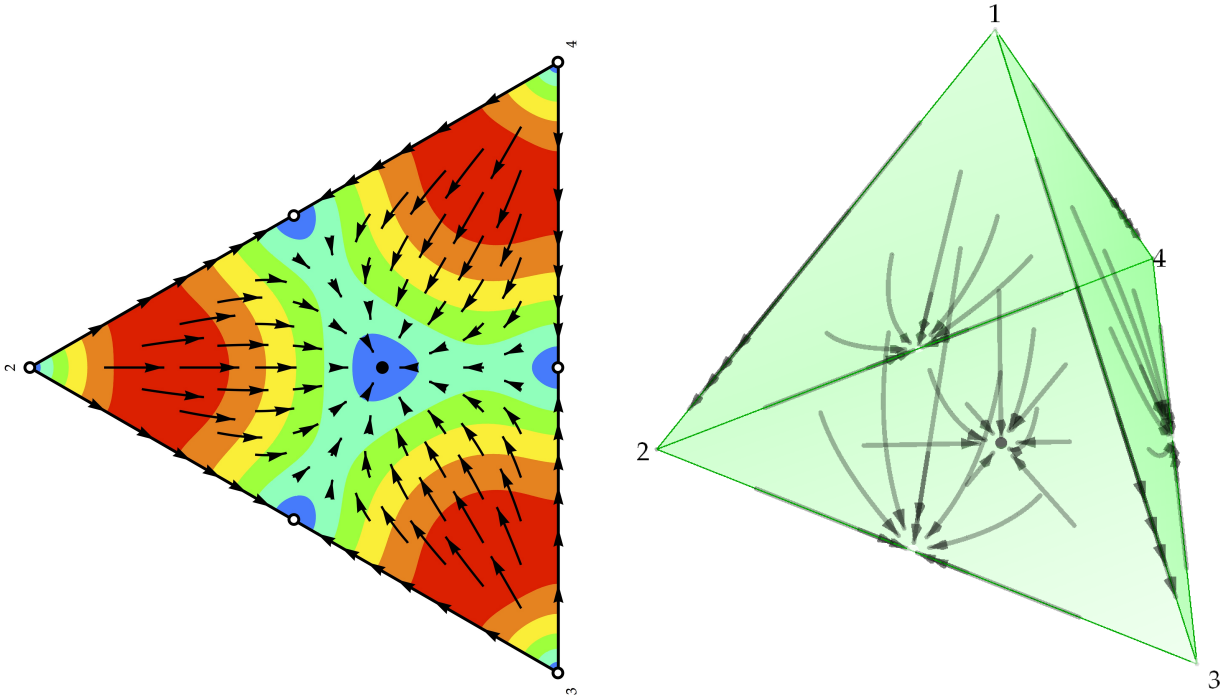
then for all $x(0) \in \operatorname{int}(\Delta)$ *it holds that* $x_i(t) \to 0$ *as* $t \to \infty$*.*

This property together with the invariance of the faces, makes the replicator dynamics a perfect case study to illustrate the results of this chapter. In what follows, we provide 6 examples, each considers a population of 4 species, the first of which vanishes in the long run (which can be shown by Proposition 3.7 or other results such as [181, Proposition 3.2]). The payoff matrices are mainly extended forms of some of those in [24]. Note that although the replicator dynamics describe the evolution of n variables, they are indeed an $(n-1)$-dimensional system due to the constraint $\sum_{i=1}^{n} x_i = 1$. So in view of the framework in Section 3.5, the vector field of the examples

are $\mathbf{C}^\ell, \ell \geq 1$, and 3-dimensional, and the compact invariant manifold $\mathcal{M}$ is two-dimensional and equal to the face $\Delta(e^2, e^3, e^4)$. As mentioned above, all e^2, e^3 and e^4 are fixed points. Moreover, the interior of every edge can be shown to have either zero, one or an infinite number of fixed points. In case an edge $\Delta(e^i, e^j), i, j \in \{2, 3, 4\}$ admits a unique fixed point in its interior, we denote the fixed point by x^{ij}. Similarly, in case the interior of the face $\Delta(e^2, e^3, e^4)$ admits a unique fixed point, we denote it by x^{234}. We determine the sign of the eigenvalues of the fixed points; the first two of each case correspond to the eigenvectors lying in $\Delta(e^2, e^3, e^4)$. For different initial states p of the simplex, we plot the flow through p with the help of the software *Dynamo* [150]. In case of a continuum of fixed points, the software only draws some of the fixed points.

Example 3.4. The payoff matrix, and the sign of the eigenvalues of the fixed points are as follows

$$\pi_{3.4} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{cases} e^2 : +, +, 0 \\ e^3 : +, +, 0 \\ e^4 : +, +, 0 \end{cases}, \begin{cases} x^{2,3} : -, +, - \\ x^{3,4} : -, +, - \\ x^{2,4} : -, +, - \end{cases}, x^{234} : -, -, -.$$



(a) Phase portrait of $f|_{\mathcal{M}}$ (b) Phase portrait of the full dynamics

Figure 3.7: Flow patterns of the replicator dynamics with the payoff matrix $\pi_{3.4}$ in Example 3.4. Circles denote fixed points. (a) Phase portrait of the face $\Delta(e^2, e^3, e^4)$ representing $\mathcal{M}$. Color temperatures are used to show motion speeds where red corresponds to the fastest and blue the slowest motion. (b) The evolution of the phase flow $\phi_t(p)$ for different values of p in the interior and on the boundary of Δ.

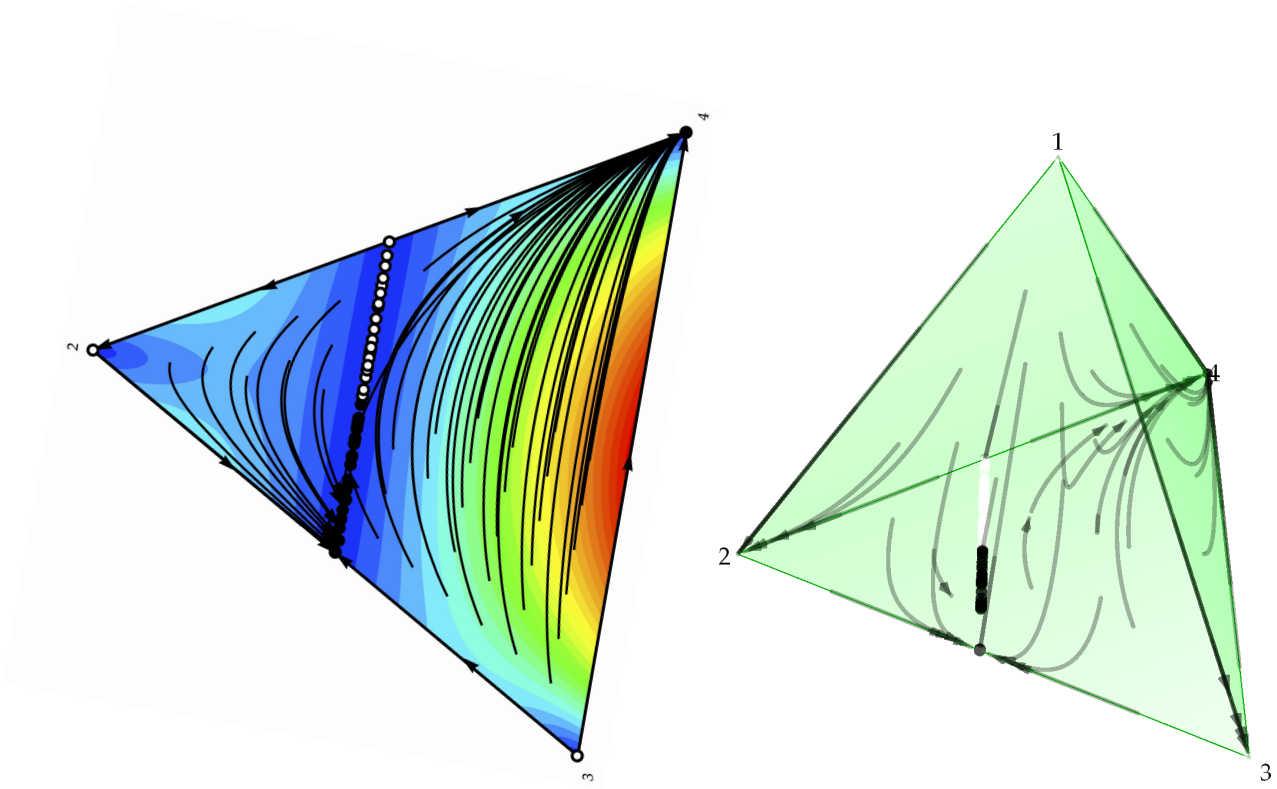
The set $\mathcal{M}$ is the face shown in Fig. 3.7-(a). There exists only one attracting set, the fixed point x^{234}, while the fixed points at the corners of $\mathcal{M}$ are semi-repelling sets. It then follows

that $\mathcal{U}_{\mathcal{A}}$ is the interior of $\mathcal{M}$ except for the inner fixed point x^{234}, i.e., $\mathcal{U}_{\mathcal{A}} = \text{int}(\mathcal{M}) - \{x^{234}\}$. Now $\cup_{\mathcal{R}' \in \mathscr{R}'} \mathcal{U}_{\mathcal{R}} = \mathcal{M} - \{e^2, e^3, e^4, x^{2,3}, x^{3,4}, x^{2,4}, x^{234}\}$. Therefore, the set $\mathcal{N}$ is given by $\mathcal{N} = \{e^2, e^3, e^4, x^{2,3}, x^{3,4}, x^{2,4}, x^{234}\}$. Moreover, from the payoff matrix π_1, every fixed point on $\mathcal{M}$ is hyperbolic under $f|_{\mathcal{M}}$, implying that Theorem 3.6 can be used. Therefore, the only possible limit sets of trajectories converging to $\mathcal{M}$ are the fixed points in $\mathcal{M}$ (see Fig. 3.7-(b)). Note that, since $\text{bd}(\mathcal{M})$ is not flow-aligned, a trajectory approaching $\mathcal{M}$ cannot converge to its boundary.

Example 3.5. The payoff matrix, and the sign of the eigenvalues of the fixed points are as follows

$$\pi_{3.5} = \begin{bmatrix} -1 & -2 & -1 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & -1 & 2 & 0 \end{bmatrix}, \quad \begin{cases} e^2 : -, +, - \\ e^3 : +, +, - \\ e^4 : -, -, - \end{cases}, \begin{cases} x \in \mathcal{X}^1 : -, 0, - \\ x \in \mathcal{X}^2 : 0, +, - \end{cases}$$

where $\mathcal{X}^1$ and X^2 denote, respectively, the stable and unstable parts of the continuum of fixed points in $\text{int}(\Delta(e^2, e^3, e^4))$.



(a) Phase portrait of $f|_{\mathcal{M}}$ (b) Phase portrait of the full dynamics

Figure 3.8: Flow patterns of the replicator dynamics with the payoff matrix $\pi_{3.5}$ in Example 3.5 Circles denote fixed points.

The set $\mathcal{N}$ is as shown in Fig. 3.9. Theorem 3.5 guarantees that the possible limit sets are contained in one of the elements of $\mathcal{N}$, that is, either a single fixed point, or the cycle shown in

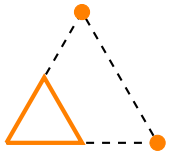


Figure 3.9: The set $\mathcal{N}$ in orange corresponding to Example 3.5. The connected elements of $\mathcal{N}$ contain the possible limit sets of a trajectory converging to $\mathcal{M}$.

Fig. 3.9 (see also Fig. 3.8). Note that the cycle is a flow-aligned closed curve constructed by two non-fixed orbits connected to a continuum of fixed points and hence is not ruled out from possible limit sets by Theorem 3.3.

Example 3.6. The payoff matrix, and the sign of the eigenvalues of the fixed points are as follows

$$\pi_{3.6} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad x \in \mathcal{X} : 0, 0, -$$

where $\mathcal{X} = \Delta(e^2, e^3, e^4)$.

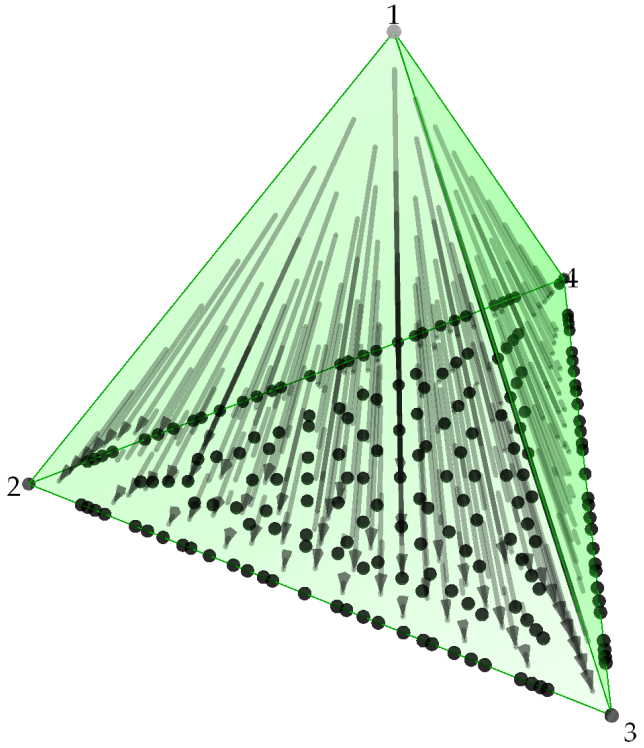


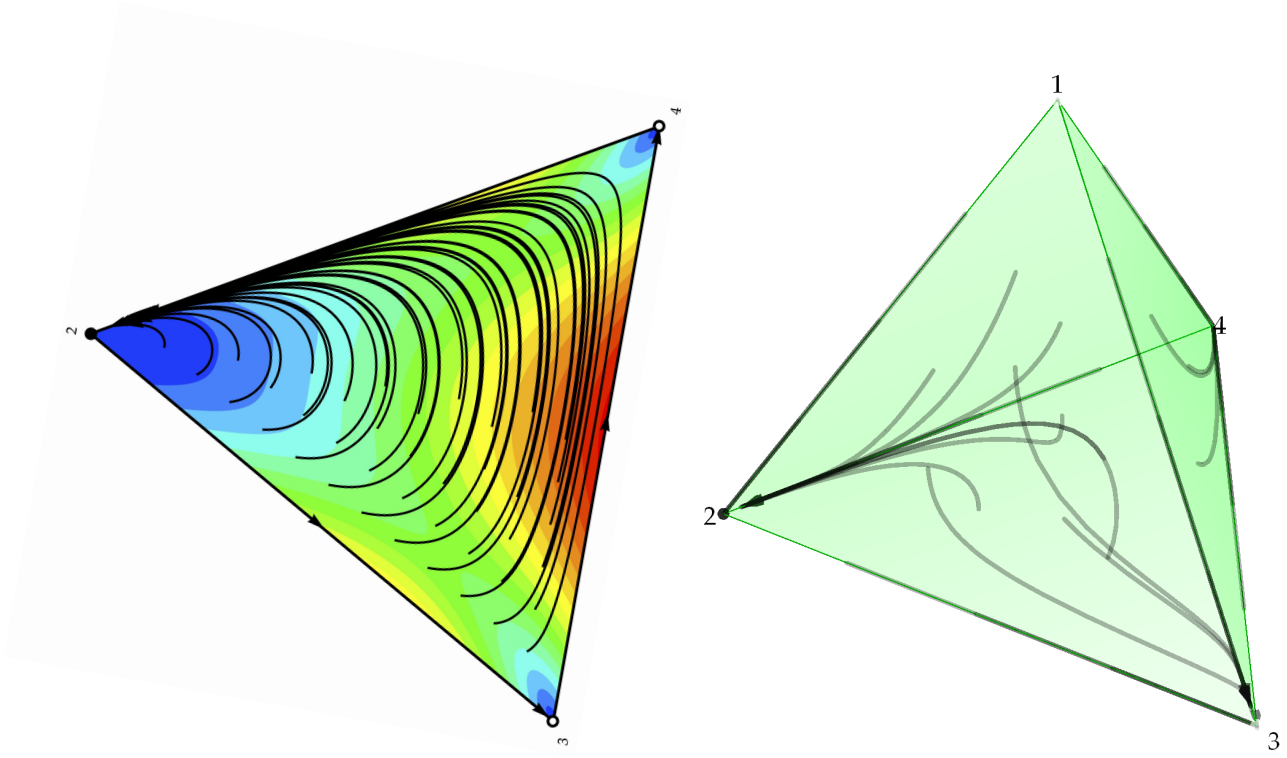
Figure 3.10: Flow patterns of the replicator dynamics with the payoff matrix $\pi_{3.8}$ in Example 3.6. Circles denote fixed points.

The set $\mathcal{M}$ corresponds to a set filled with fixed points, and $\mathcal{N} = \mathcal{M}$. This means that the limit set of a trajectory converging to $\mathcal{M}$ can, in principle, be any compact subset of $\mathcal{M}$, in view

of Theorem 3.5. In this case, however, since every point $p \in \mathcal{M}$ has an associated transverse stable eigenspace, we can conclude that every trajectory converging to $\mathcal{M}$ converges to one and only one fixed point on $\mathcal{M}$ (see Fig. 3.10).

Example 3.7. The payoff matrix, and the sign of the eigenvalues of the fixed points are as follows

$$\pi_{3.7} = \begin{bmatrix} -1 & -1 & 0 & -2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{cases} e^2 : 0, 0, 0 \\ e^3 : -, +, 0 \\ e^4 : -, +, - \end{cases},$$



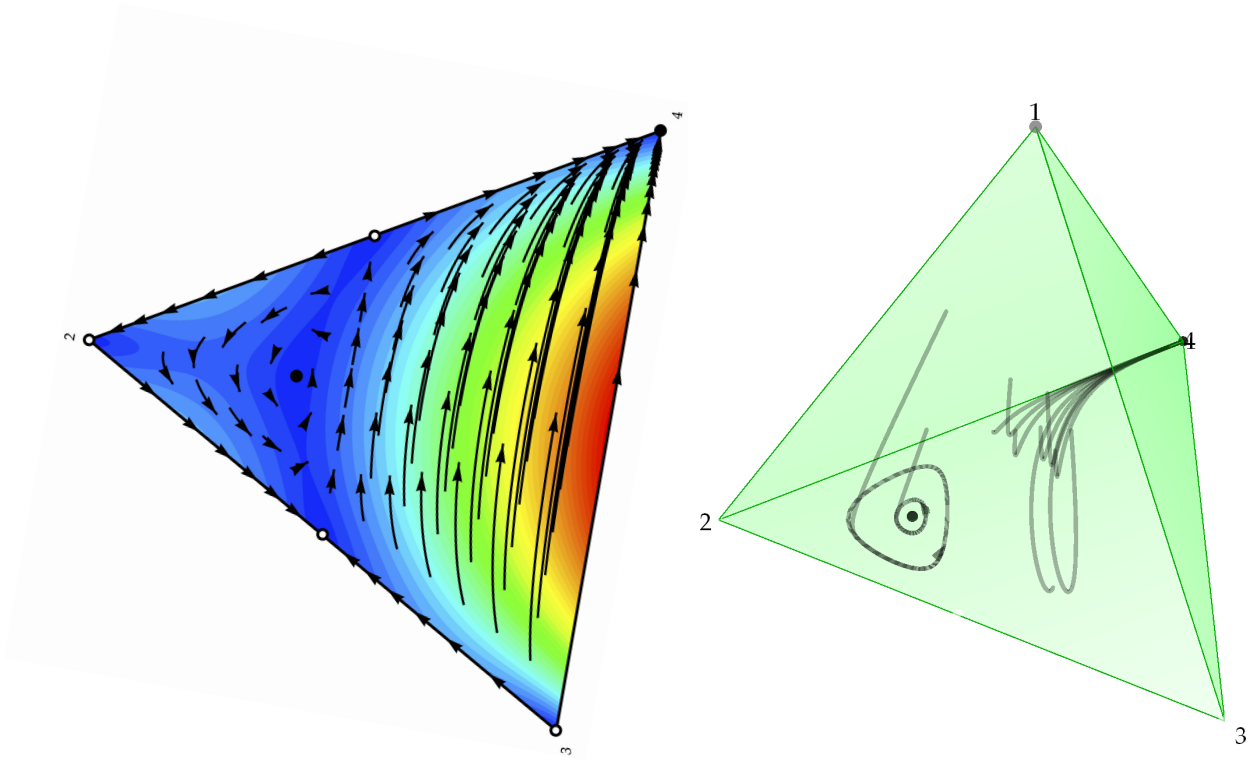
(a) Phase portrait of $f|_{\mathcal{M}}$ (b) Phase portrait of the full dynamics

Figure 3.11: Flow patterns of the replicator dynamics with the payoff matrix $\pi_{3.7}$ in Example 3.7. Circles denote fixed points.

As shown in Fig. 3.11-(a), every trajectory in $\mathcal{M}$ is a homoclinic orbit. Moreover, there is no (semi-)trapping region, and thus, no (semi-)attracting nor (semi-)repelling sets. This implies that $\mathcal{N} = \mathcal{M}$ and then, any compact, invariant subset of $\mathcal{M}$ cannot be ruled out by Theorem 3.5 from being a limit set of a trajectory converging to $\mathcal{M}$. See Fig. 3.11-(b) for some trajectories converging to $\mathcal{M}$.

Example 3.8. The payoff matrix, and the sign of the eigenvalues of the fixed points are as follows

$$\pi_{3.8} = \begin{bmatrix} -1 & -4 & -4 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & -1 & 3 & 0 \end{bmatrix}, \quad \begin{cases} e^2 : -,+,- \\ e^3 : +,+,- \\ e^4 : -,-,- \end{cases}, \begin{cases} x^{2,3} : -,+,- \\ x^{2,4} : -,+,-, \end{cases} \quad x^{2,3,4} : \pm\frac{\sqrt{3}}{5}j, -.$$



(a) Phase portrait of $f|_{\mathcal{M}}$ (b) Phase portrait of the full dynamics

Figure 3.12: Flow patterns of the replicator dynamics with the payoff matrix $\pi_{3.8}$ in Example 3.8. Circles denote fixed points.

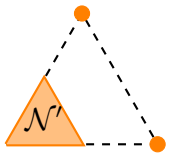


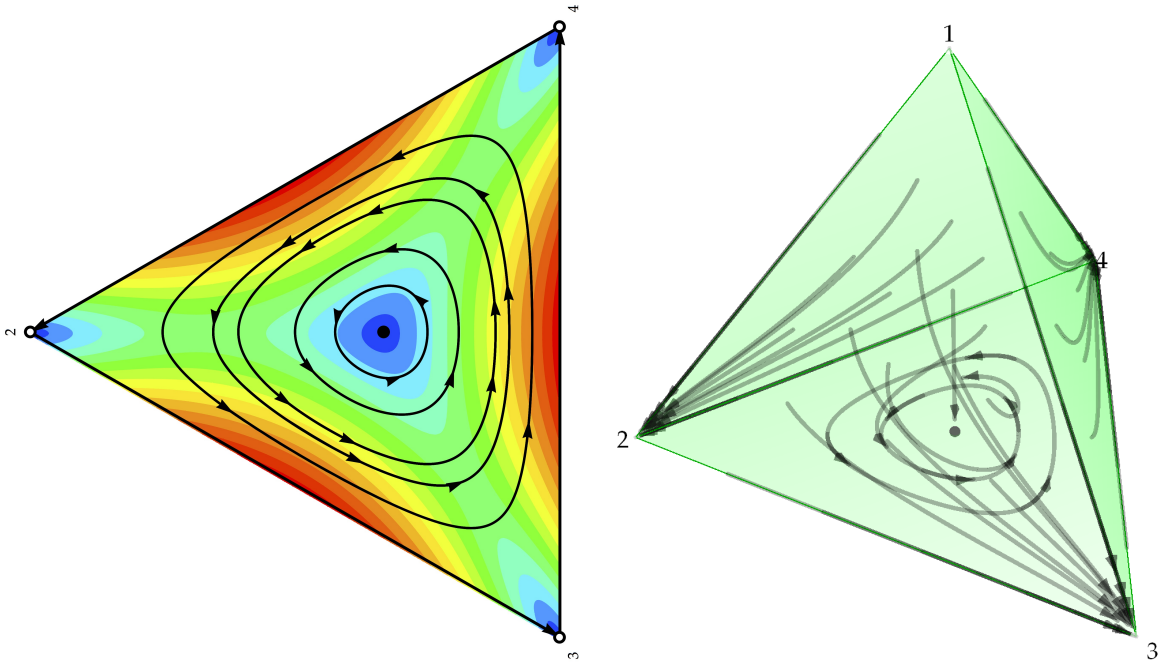
Figure 3.13: The set $\mathcal{N}$ corresponding to Example 3.8. The connected elements of $\mathcal{N}$ contain the possible limit sets of a trajectory converging to $\mathcal{M}$. In this case, and in contrast to Example 3.5, the cycle shown in the figure, together with its interior, is a possible limit set of a trajectory approaching $\mathcal{M}$

An analysis of $f|_{\mathcal{M}}$, depicted in Fig. 3.12-(a), shows that $\mathcal{N}$ is as given in Fig. 3.13. It follows from Theorem 3.5 that the possible limit sets are any of the fixed points in $\mathcal{M}$ or a compact,

invariant subset of the $\mathcal{N}'$ shown in Fig. 3.13. As shown in Fig. 3.12-(b), the trajectories indeed converge to the aforementioned limit sets.

Example 3.9. The payoff matrix, and the sign of the eigenvalues of the fixed points are as follows.

$$\pi_{3.9} = \begin{bmatrix} 0 & -.5 & -.5 & -.5 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad \begin{cases} e^2 : -, +, - \\ e^3 : -, +, - \\ e^4 : -, +, - \end{cases} \quad , x^{2,3,4} : \pm .6j, -$$



(a) Phase portrait of $f|_{\mathcal{M}}$ (b) Phase portrait of the full dynamics

Figure 3.14: Flow patterns of the replicator dynamics with the payoff matrix $\pi_{3.9}$ in Example 3.9. Circles denote fixed points.

There are no attracting, repelling, semi-attracting or semi-repelling sets of $f|_{\mathcal{M}}$ (see Fig. 3.14-(a)). Therefore $\mathcal{M} = \mathcal{N}$. This implies that no compact, invariant subset of $\mathcal{M}$ may be ruled out by Theorem 3.5 from the limit set of a trajectory converging to $\mathcal{M}$. Figure 3.14-(b) shows examples of those compact limit sets, among which are fixed points and periodic orbits.

3.8 Concluding remarks

We have studied the limit sets of trajectories converging to a compact invariant curve Γ or a compact invariant manifold $\mathcal{M}$. Our results complement some fundamental theorems such as those found in [167, 22]. Our main contribution is the description of the limit set of a trajectory, which converges to $\mathcal{M}$, in terms of the invariant sets contained in $\mathcal{M}$. We have shown that there

exist compact invariant sets $\mathcal{N}_i$ where the ω limit set is contained. For particular applications, our results shed light on the topological properties of ω limit sets in some special dimension. For example, if $\mathcal{M}$ is 1-dimensional and in the form of a simple open curve, we show that the ω limit set is either a fixed-point or a continuum of fixed points; if $\mathcal{M}$ is 2-dimensional and has only hyperbolic fixed points, then the ω limit set is either an isolated fixed point, or a compatibly oriented closed curve. Our results can be applied to some biological models such as population dynamics, where in a series of examples, we are able to rigorously identify the ω limit sets. Thus we provide proof on the fate of certain species by only looking at the invariant sets of the phase space.

Our results are also aimed towards extending the Poincaré-Bendixson theorem to arbitrary dimensions. Although the current state-of-the-art is far away from a complete classification of ω limit sets in dimensions higher than 2, our main result, i.e., Theorem 3.5, already refines the possibilities by providing a constructive way to discern invariant subsets containing ω limit sets in arbitrary dimensions. For example, consider a compact planar $\mathcal{M}$ embedded in an invariant, compact set $\mathcal{D} \subset \mathbb{R}^3$. For a point $p \in \mathcal{D}$, we are interested in $\omega(p)$. If $\varphi_t(p) \to \mathcal{M}$, then Theorem 3.5 implies that $\omega(p)$ is contained in an isolated and compact candidate limit set $\mathcal{N}_i$. This already provides a sort of classification of the possible limit sets. If, in addition, the vector field restricted to $\mathcal{M}$ has only hyperbolic fixed points, then Theorem 3.6 provides a complete classification of $\omega(p)$.

Several future research topics can be specified, the most immediate one is the extension of Theorem 3.5 to consider $\mathcal{M}$ as any invariant set and not necessarily a manifold. Another is to refine Theorem 3.5 in case where $\mathcal{M}$ has some particular properties, such as the case where $\mathcal{M}$ has non-hyperbolic fixed points. Finally, one can explore the application to the controlled dynamics, where the possible limit sets of the state of the open-loop system is known, but we one would like to avoid some due to their undesired properties such as instability. Then Theorem 3.5 suggests that it is sufficient to change only the vector field close to those undesired limit sets, and not the whole domain of the state space (since the state will surely converge to one of the limit sets).

Chapter 4

Global convergence for replicator dynamics of repeated snowdrift games

To understand the emergence and sustainment of cooperative behavior in interacting collectives, we perform global convergence analysis for the evolutionary replicator dynamics of a large, well-mixed population of individuals playing a repeated snowdrift game with four typical strategies, which are always cooperate (ALLC), tit-for-tat (TFT), suspicious tit-for-tat (STFT) and always defect (ALLD). The dynamical system model in focus is a three-dimensional ODE system that is parameterized by the payoffs of the base game. We demonstrate how the asymptotic behavior of the system changes as the mutual cooperation payoff changes, and in particular show that for the full range of payoffs, every trajectory of the system converges to an equilibrium point. The convergence results highlight three findings that are of particular importance for understanding the cooperation mechanisms among self-interested agents playing repeated snowdrift games. First, the inclusion of TFT- and STFT-players, the two types of conditional strategy players in the game, increases the share of cooperators of the overall population compared to the situation when the population consists of only ALLC- and ALLD-players. This confirms findings in biology and sociology that reciprocity may promote cooperation in social collective actions, such as reducing traffic jams and division of labors, where each individual may gain more to play the opposite of what her opponent chooses. Second, surprisingly enough, regardless of the payoffs, there always exists a set of initial conditions under which ALLC players show up in the long run, which does not hold for all the other three types of players. So an ALLC-player, although perceived as the one that can be easily taken advantage of in snowdrift games, has certain endurance in the long run. Third, the parametric framework makes it possible to actually control the final population shares, a challenging topic in population dynamics, by tuning the payoffs of the base game.

4.1 Introduction

After presenting the mathematical results in Chapters 2 and 3, now we proceed to one of the main sub-problems in this thesis, that is how repetition of a game played by the individuals of a population can help the promotion of cooperation in that population. Game theory provides a framework for studying various control problems such as robust control, distributed control and

optimization for traffic systems, communication networks and multi-agent systems in general; in this context, the different types of games that have been modeled and analyzed in the literature include potential games [97, 39, 89, 98, 90], stochastic games [10, 32, 182], constrained games [11], repeated games [96, 155], matrix games [26], networked games [58], and others [47, 162, 173, 95, 113, 51]. More recetnly, evolutionary game theory has gained more attention since it is a powerful tool in understanding the evolution of cooperation among selfish individuals as reported by biologists, sociologists, economists, etc [176, 115, 149, 181, 139]. Researchers have found that network topology [141], phenotypic interactions [73, 138], punishment [175], population heterogeneity [136], as well as other components in game setups can all affect the success of cooperators in face of defectors. One stimulating mechanism for the evolution of cooperation that is generally believed to promote cooperation, especially in human societies [177], is *direct reciprocity* [172]. This mechanism is captured by *repeated games* where individuals play a base game repeatedly and can base their action in each round of the game on that of the opponent in the previous round, resulting in *reactive strategies*.

Perhaps the most typical reactive strategy is the simple yet successful *tit-for-tat (TFT)* strategy where the player starts with cooperation and cooperates if the opponent cooperated and defects if the opponent defected in the last round. A more defective version of the strategy is the *suspicious tit-for-tat ($STFT$)* strategy which is the same as TFT except that the player starts with defection. In addition to these conditional strategies, there are two unconditional ones which are the two extreme strategies in repeated 2-strategy games: *always-cooperate ($ALLC$)* and *always-defect ($ALLD$)*. While much research has been carried out to investigate the performance of different reactive strategies under the prisoner's dilemma game, the cornerstone of game theory, [114, 69, 63, 56, 93, 68], less has been devoted to the anti-coordination snowdrift game [134, 86, 178] despite the fact that the snowdrift game captures many behavioral patterns that cannot be well-modeled by the prisoner's dilemma game [61]. Moreover, the existing results on the snowdrift game are mainly experimental or simulation based. For example, in [86], based on human experiments the authors postulate that iterated snowdrift games can explain high levels of cooperation among non-relative humans. However, few mathematical statements have been constructed to support such claims.

The performance of different reactive strategies also remains an open problem. Usually the strategies are compared using 2-strategy games, e.g., the two famous competitions conducted by Axelord [16, 17] where strikingly, the simple TFT was placed first in both (note that although TFT is known to be successful mostly in the repeated prisoner's dilemma, it has also been reported to be successful in the repeated snowdrift game [86, 42]). The situation would be different if more than two strategies could be played in the game. Then the best strategy can be decided by natural selection, which is captured by evolutionary dynamics such as the well-known *replicator dynamics* [20, 27, 28, 41]. Due to nonlinearity, the replicator dynamics, however, may exhibit quite complex behaviors. Indeed, except for a few cases [40] [188], analytical analysis is restricted to only those modeled by planar dynamical systems [25]. This makes the performance investigation of more than three reactive strategies generally challenging under the replicator dynamics.

We address both of the above issues in this chapter. While considering the snowdrift game as the base game, we study the evolution of a large population of individuals playing the four just mentioned strategies, $ALLC$, TFT, $STFT$ and $ALLD$, under the replicator dynamics. We consider a completely parameterized payoff matrix with an arbitrary number of repetitions of the

base game and present rigorous proofs for the convergence analysis of the resulting 3-dimensional dynamics. In particular, by studying the dynamics of two crucial ratios of the state variables, we show that each trajectory of the system converges to an equilibrium point, excluding the possibility of a limit cycle or chaotic behaviors. Our convergence analyses shed light on the social dilemma in the snowdrift game, that is why selfish individuals cooperate while they earn more if they defect against their cooperative opponents. This is done by showing that first of all, even in the presence of the very defective strategy $ALLD$, for some range of payoffs and initial population portions, the population evolves to the state where all mutually cooperate. In other words, natural selection disfavors individuals playing $ALLD$ and instead chooses those playing more cooperative strategies such as TFT and even $ALLC$. Secondly, the convergence results postulate that among the four types of players, $ALLC$s are surprisingly the best in terms of survival and appearance in the long run, explaining why selfish individuals may repeatedly cooperate in a snowdrift social context. As a second contribution, due to the parametric framework we provide, our stability analysis can be used to actually control the final state of the replicator dynamics. By tuning the parameters, one can control the final population portions of individuals playing the reactive strategies. Moreover, for populations initially having four co-existing types of players, by comparing those final states in which one or two types of players die out to those with all four, it becomes clear how adding a third or fourth strategy can change the final population state. These results lead to addressing the crucial question of *how to control portions of different types of individual in a decision-making population?*

The rest of the chapter is organized as follows. In Section 4.2, we describe the replicator dynamics for repeated snow drift games with the above four reactive strategies. In Section 4.3, we provide the global convergence results and discuss their implications on the success of the strategies. We end with the concluding remarks in Section 4.4.

4.2 Problem formulation

We consider an infinitely large, well-mixed population of individuals that are playing repeated games over time. Each game has two players with two pure strategies: one is to cooperate, denoted by C, and the other to defect, denoted by D, and the payoffs of the game, described by the following payoff matrix, are symmetric to both players

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array}, \tag{4.1}$$

where R, S, T and P are real numbers and sometimes in the literature are called the reward, sucker's payoff, temptation and punishment respectively. We call this two-player, symmetric game, the *base game* and denote it by G. When the payoffs of the game satisfy

$$T > R > S > P, \tag{4.2}$$

the game is called a *snowdrift game* (also known as the *hawk-dove* or the *chicken game*). The game has two Nash equilibria in pure strategies, both of which correspond to the situation when the two

players play opposite strategies, and for this reason such a game is also called an *anti-coordination* game, often used to study how players may contribute to the accomplishment of a common task. In this study, we are particularly interested in the case in which individuals play the game repeatedly over time and adjust their strategies according to what their opponents have played in the past. Formally, a *repeated game, denoted by* G^m, $m \geq 2$, with *reactive strategies* is constructed from the base game G by repeating it for m rounds, and limiting a player's choice of strategies in the current round to be based on the opponent's choice in the previous round. In fact, a reactive strategy s can always be represented by the triple (p, q, r), where p is the probability of cooperating in the first round, and q (respectively r) is the probability of cooperating if the opponent has cooperated (respectively defected) in the previous round. We consider the following strategies:

- *always-cooperate (ALLC)*, $(1, 1, 1)$: always cooperates;
- *tit-for-tat (TFT)*, $(1, 1, 0)$: cooperates in the first round, and then chooses what the opponent did in the previous round;
- *suspicious-tit-for-tat (STFT)*, $(0, 1, 0)$: defects in the first round, and then chooses what the opponent did in the previous round;
- *always-defect (ALLD)*, $(0, 0, 0)$: always defects.

When two players play the repeated game G^m, the payoffs for the reactive strategies can be calculated every m rounds, leading to the payoff matrix $A := [a_{ij}]$ defined by

$$A = \begin{array}{c} \\ ALLC \\ TFT \\ STFT \\ ALLD \end{array} \begin{array}{c} \begin{array}{cccc} ALLC & TFT & STFT & ALLD \end{array} \\ \begin{pmatrix} mR & mR & S+(m-1)R & mS \\ mR & mR & \lceil \frac{m}{2} \rceil S + \lfloor \frac{m}{2} \rfloor T & S+(m-1)P \\ T+(m-1)R & \lceil \frac{m}{2} \rceil T + \lfloor \frac{m}{2} \rfloor S & mP & mP \\ mT & T+(m-1)P & mP & mP \end{pmatrix} \end{array}.$$

To illustrate how the matrix A is obtained, we take the match between TFT and $ALLD$ as an example. In round one, the TFT player cooperates and the $ALLD$ player defects, so their payoffs according to (4.1) are S and T respectively. From round two, both TFT and $ALLD$ players defect and hence receive P. So over time the payoffs for the TFT player are $S, P, P, \ldots$ while those for the $ALLD$ player are $T, P, P, \ldots$. Summing up the payoffs over the m rounds, one obtains the entries of a_{23} and a_{32} in A. Hence, the repeated game G^m can be taken as a normal, symmetric two-player game with the payoff matrix A and with the pure-strategy set $\{ALLC, TFT, STFT, ALLD\}$. Having clarified how a pair of individuals play games with each other, we now describe the evolutionary dynamics of the whole population. Towards this end, we introduce *replicator dynamics*, which is a standard model from evolutionary game theory [181, 149].

Let $0 \leq x_i(t) \leq 1, i = 1, 2, 3$ and 4, denote the population shares at time t of those individuals playing the pure strategies $ALLC$, TFT, $STFT$ and $ALLD$ respectively. Since the four types of players constitute the whole population, it follows that for all t, $\sum_{i=1}^{4} x_i = 1$. Define the population vector

$$x := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top.$$

Then $x \in \Delta$ where Δ is the 4-dimensional simplex defined by

$$\Delta := \left\{ z \,|\, z \in \mathbb{R}^4, z_i \geq 0, i = 1, \ldots, 4, \sum_{i=1}^{4} z_i = 1 \right\}. \tag{4.3}$$

We use the unit vectors at the vertices of the simplex

$$p_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, p_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

to represent the population vectors corresponding to all $ALLC$ players, all TFT players, all $STFT$ players and all $ALLD$ players respectively. Then the evolutions of x_i, $i = 1, \ldots, 4$, are described by the replicator dynamics [181, 149]

$$\dot{x}_i = [u(p_i, x) - u(x, x)]x_i, \tag{4.4}$$

where $u(\cdot, \cdot)$ is the *utility function* defined by

$$u(x, y) = x^\top Ay \text{ for } x, y \in \Delta$$

determining the fitness of a player. In essence, (4.4) indicates that in an evolutionary process, the reproduction rate of the strategy-i players is proportional to the difference between the fitness of strategy-i players $u(p_i, x)$ and the average population fitness $u(x, x)$ as a consequence of the fact that the more payoff an individual acquires when playing against its opponents, compared to the average payoff of the whole population, the more new offspring proportionally it produces.

We perform global convergence analysis of the replicator dynamics (4.4). More specifically, for any given initial condition $x(0) \in \Delta$, we aim to determine the asymptotic state of $x(t)$ for (4.4).

4.3 Global convergence result

The main results of this chapter are presented in this section. First we find the equilibrium points of the system. Then for the convergence results, we divide the analysis into several parts using the notion of *face* defined as follows. A face of the simplex is the convex hull of a non-empty subset $\mathcal{H}$ of $\{p_1, p_2, p_3, p_4\}$, and is denoted by $\Delta(\mathcal{H})$. For simplicity, we remove the braces when $\mathcal{H}$ is represented by its members. For example, the face $\Delta(p_1, p_3, p_4)$ is the convex hull of $\mathcal{H} = \{p_1, p_3, p_4\}$. When $\mathcal{H}$ is proper, $\Delta(\mathcal{H})$ is called a *boundary face*. Lemma 1.3 enables us to analyze the evolution of a trajectory starting from $\text{bd}(\Delta)$ separately from that starting from $\text{int}(\Delta)$.

We start with analyzing the boundary of the simplex. However, the boundary of the simplex itself consists of the four planar faces $\Delta(p_1, p_2, p_3)$, $\Delta(p_1, p_2, p_4)$, $\Delta(p_1, p_3, p_4)$ and $\Delta(p_2, p_3, p_4)$. Because of Lemma 1.3, we can also analyze the dynamics (4.4) on each of these faces separately. Yet again, the boundary of each of these planar faces consists of three one-dimensional faces known as the *edges* of the simplex. For example, the boundary of the face $\Delta(p_1, p_2, p_3)$ consists of the edges $\Delta(p_1, p_2)$, $\Delta(p_1, p_3)$ and $\Delta(p_2, p_3)$. On the other hand, each of the edges are also invariant in

view of Lemma 1.3. Therefore, we study separately trajectories starting from an edge and those starting from the interior of a planar face. Then we proceed to the interior of the simplex.

To simplify the analysis, we carry out on the matrix A some operations that preserve the dynamics (4.4). Subtracting mR from the entries of the first and second columns, and mS from the entries of the third and fourth columns of A, we acquire the following matrix

$$A' := [a'_{ij}] = \begin{bmatrix} 0 & 0 & S+(m-1)R-mP & m(S-P) \\ 0 & 0 & \lceil \frac{m}{2} \rceil S + \lfloor \frac{m}{2} \rfloor T - mP & S-P \\ T-R & \lceil \frac{m}{2} \rceil T + \lfloor \frac{m}{2} \rfloor S - mR & 0 & 0 \\ m(T-R) & T+(m-1)P-mR & 0 & 0 \end{bmatrix}. \tag{4.5}$$

In view of Lemma 1.3, the dynamics (4.4) are unchanged with A' in place of A. Since A' is more structured with zero block matrices, in what follows we focus on A' instead of A.

4.3.1 Equilibrium points

To determine the equilibria of the system, we first look for those on the boundary of the simplex, and then for those in the interior.

boundary equilibrium points

Let Δ^o and Δ^{oo} denote the set of equilibrium points of the replicator dynamics (4.4) that belong to Δ and $\mathrm{bd}(\Delta)$, respectively. Depending on the payoffs, Δ^{oo} will be a combination of the unit vectors p_1, p_2, p_3, p_4, the vectors

$$x^{14} = \begin{bmatrix} \frac{S-P}{S-P+T-R} \\ 0 \\ 0 \\ \frac{T-R}{S-P+T-R} \end{bmatrix}, x^{23} = \begin{bmatrix} 0 \\ \frac{\lceil \frac{m}{2} \rceil S + \lfloor \frac{m}{2} \rfloor T - mP}{m(T+S-P-R)} \\ \frac{\lceil \frac{m}{2} \rceil T + \lfloor \frac{m}{2} \rfloor S - mR}{m(T+S-P-R)} \\ 0 \end{bmatrix}, x^{13} = \begin{bmatrix} \frac{S+(m-1)R-mP}{T+S+(m-2)R-mP} \\ 0 \\ \frac{T-R}{T+S+(m-2)R-mP} \\ 0 \end{bmatrix}, x^{24} = \begin{bmatrix} 0 \\ \frac{S-P}{T+S+(m-2)P-mR} \\ 0 \\ \frac{T+(m-1)P-mR}{T+S+(m-2)P-mR} \end{bmatrix},$$

and the sets

$$\begin{aligned} \mathcal{X}^{12} &= \{\alpha p_1 + (1-\alpha)p_2 : \alpha \in [0,1]\}, \\ \mathcal{X}^{34} &= \{\alpha p_3 + (1-\alpha)p_4 : \alpha \in [0,1]\}, \\ \mathcal{X}^{123} &= \{x \in \mathrm{int}(\Delta(p_1,p_2,p_3)) \,|\, a'_{31}x_1 + a'_{32}x_2 - a'_{13}x_3 = 0\} \end{aligned}$$

where a'_{ij}'s are the entires of A' defined in (4.5). Here, the superscript ij in x^{ij} (resp. $\mathcal{X}^{ij}$) simply means that x^{ij} (resp. $\mathcal{X}^{ij}$) belongs to the edge $\Delta(p_i, p_j)$. The following proposition determines Δ^{oo}.

Proposition 4.1. *Assume* (4.2) *holds. It follows that*

1. *if* $S < R < \frac{T+(m-1)P}{m}$, *then*

$$\Delta^{oo} = \mathcal{X}^{12} \cup \{x^{13}, x^{14}, x^{23}, x^{24}\} \cup \mathcal{X}^{34};$$

2. *if* $\frac{T+(m-1)P}{m} \leq R < \frac{T+S}{2}$, *or if* $m = 2n+1, n \geq 1$ *and* $\frac{T+S}{2} < R < \frac{(n+1)T+nS}{2n+1}$, *then*

$$\Delta^{oo} = \mathcal{X}^{12} \cup \{x^{13}, x^{14}, x^{23}\} \cup \mathcal{X}^{34};$$

3. *if* $m = 2n+1, n \geq 1$ *and* $R = \frac{T+S}{2}$, *then*

$$\Delta^{oo} = \mathcal{X}^{12} \cup \{x^{13}, x^{14}, x^{23}\} \cup \mathcal{X}^{34} \cup \mathcal{X}^{123};$$

4. *if* $m = 2n, n \geq 1$ *and* $R = \frac{nT+(n-1)S}{2n-1}$, *then*

$$\Delta^{oo} = \mathcal{X}^{12} \cup \{x^{13}, x^{14}\} \cup \mathcal{X}^{34} \cup \mathcal{X}^{123};$$

5. *if* $\max\left\{\frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1}, \frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m}\right\} < R < T$, *or if* $m = 2n, n \geq 1$ *and* $\frac{T+S}{2} \leq R < \frac{nT+(n-1)S}{2n-1}$, *then*

$$\Delta^{oo} = \mathcal{X}^{12} \cup \{x^{13}, x^{14}\} \cup \mathcal{X}^{34}.$$

For the proof, we need to take a closer look at the payoff matrix A'. The order in the magnitudes of the entries in each column of A', clarified in the following lemma, proves useful both in the determination of the equilibria and the asymptotic behavior of the replicator dynamics (4.4).

Lemma 4.1. *Assume* (4.2) *holds. Consider the payoff matrix* A' *and denote the maximum positive, positive, negative and minimum negative entries of each column by '++', '+' , '−' and '−−', respectively. Then* A' *has the following sign structure*

1. $\begin{bmatrix} 0 & 0 & + & ++ \\ 0 & 0 & ++ & + \\ + & ++ & 0 & 0 \\ ++ & + & 0 & 0 \end{bmatrix}$ *when* $S < R < \frac{T+(m-1)P}{m}$;

2. $\begin{bmatrix} 0 & 0 & + & ++ \\ 0 & 0 & ++ & + \\ + & ++ & 0 & 0 \\ ++ & 0,- & 0 & 0 \end{bmatrix}$ *when* $\frac{T+(m-1)P}{m} \leq R < \frac{T+S}{2}$;

3. $\begin{bmatrix} 0 & 0 & ++ & ++ \\ 0 & 0 & ++ & + \\ + & ++ & 0 & 0 \\ ++ & - & 0 & 0 \end{bmatrix}$ *when* $m = 2n+1, n \geq 1$, *and* $R = \frac{T+S}{2}$;

4. $\begin{bmatrix} 0 & 0 & ++ & ++ \\ 0 & 0 & + & + \\ + & ++,0 & 0 & 0 \\ ++ & - & 0 & 0 \end{bmatrix}$ *when* $m = 2n+1, n \geq 1$, *and* $\frac{T+S}{2} < R \leq \frac{(n+1)T+nS}{2n+1}$;

5. $\begin{bmatrix} 0 & 0 & + & ++ \\ 0 & 0 & ++ & + \\ + & 0,- & 0 & 0 \\ ++ & -- & 0 & 0 \end{bmatrix}$ *when* $m = 2n, n \geq 1$, *and* $\frac{T+S}{2} \leq R < \frac{nT+(n-1)S}{2n-1}$;

6. $\begin{bmatrix} 0 & 0 & ++ & ++ \\ 0 & 0 & ++ & + \\ + & - & 0 & 0 \\ ++ & -- & 0 & 0 \end{bmatrix}$ *when* $m = 2n, n \geq 1$, *and* $R = \frac{nT+(n-1)S}{2n-1}$;

7. *and* $\begin{bmatrix} 0 & 0 & ++ & ++ \\ 0 & 0 & + & + \\ + & - & 0 & 0 \\ ++ & -- & 0 & 0 \end{bmatrix}$ *when* $\max\left\{\frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1}, \frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m}\right\} < R < T$.

Here, when an entry takes both 0 *and one other sign (separated by a comma),* 0 *takes place if the equality sign of the* R *condition holds, and otherwise the other sign is valid.*

Proof. The sign of the elements of A' are determined by (4.2). First note that $T > R$ implies $a'_{31} > 0$. On the other hand, since $m \geq 2$, we have that $a'_{41} > a'_{31} > 0$. Hence, due to the fact that the third and fourth entries of the last column of A' are zero, a'_{41} and a'_{31} are denoted by '++' and '+', respectively. Similarly $S > P$ implies $a'_{14} > a'_{24} > 0$ and hence a'_{14} and a'_{24} are denoted by '++' and '+', respectively. Since $T, S > P$ implies

$$\lceil\frac{m}{2}\rceil S + \lfloor\frac{m}{2}\rfloor T > \lceil\frac{m}{2}\rceil P + \lfloor\frac{m}{2}\rfloor P \Rightarrow \lceil\frac{m}{2}\rceil S + \lfloor\frac{m}{2}\rfloor T > mP,$$

it follows that $a'_{23} > 0$. Additionally,

$$R > P,\ S > P \Rightarrow (m-1)R + S > mP,$$

which implies $a'_{13} > 0$. Similarly $T, S > P$ yields

$$\begin{aligned} &T + \lceil\tfrac{m-2}{2}\rceil T + \lfloor\tfrac{m-2}{2}\rfloor S + S > T + \lceil\tfrac{m-2}{2}\rceil P + \lfloor\tfrac{m-2}{2}\rfloor P + P \\ &\Rightarrow \lceil\tfrac{m}{2}\rceil T + \lfloor\tfrac{m}{2}\rfloor S > T + (m-2)P + P = T + (m-1)P. \end{aligned}$$

Hence, $a'_{32} > a'_{42}$. It remains to determine the signs of a'_{42} and a'_{32} and also the ordering of a'_{13} and a'_{23}. Since $m \geq 2$, devision by $m-1$ is valid, and hence the following hold

$$a'_{42} > 0 \iff R < \frac{T + (m-1)P}{m}, \tag{4.6}$$

$$a'_{32} > 0 \iff R < \frac{\lceil\frac{m}{2}\rceil T + \lfloor\frac{m}{2}\rfloor S}{m}, \tag{4.7}$$

$$a'_{23} > a'_{13} \iff R < \frac{\lceil\frac{m-2}{2}\rceil S + \lfloor\frac{m}{2}\rfloor T}{m-1}. \tag{4.8}$$

The average of $T, \underbrace{P, \ldots, P}_{m-1}$ is less than both the average of $T, \underbrace{T, \ldots, T}_{\lceil\frac{m}{2}\rceil - 1}, \underbrace{S, \ldots, S}_{\lfloor\frac{m}{2}\rfloor}$ and the average of $T, \underbrace{T, \ldots, T}_{\lfloor\frac{m}{2}\rfloor - 1}, \underbrace{S, \ldots, S}_{\lceil\frac{m-2}{2}\rceil}$. Thus,

$$\frac{T + (m-1)P}{m} < \frac{\lceil\frac{m}{2}\rceil T + \lfloor\frac{m}{2}\rfloor S}{m}, \frac{\lceil\frac{m-2}{2}\rceil S + \lfloor\frac{m}{2}\rfloor T}{m-1}.$$

Hence, when (4.6) holds, so do (4.7) and (4.8). This proves the first case of the lemma. Now we compare $\frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1}$ and $\frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m}$. In general, it holds that

$$\frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1}=\begin{cases}\dfrac{(n-1)S+nT}{2n-1} & m=2n\\ \dfrac{nS+nT}{2n} & m=2n+1\end{cases},$$

$$\frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m}=\begin{cases}\dfrac{nT+nS}{2n} & m=2n\\ \dfrac{(n+1)T+nS}{2n+1} & m=2n+1\end{cases}.$$

Due to the fact that $T>S$, we obtain

$$\begin{aligned}\frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m}&<\frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1} && m=2n,\\ \frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1}&<\frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m} && m=2n+1.\end{aligned}\tag{4.9}$$

Hence,

$$\min\left\{\frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m},\frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1}\right\}=\frac{T+S}{2}.\tag{4.10}$$

The above equation results in cases 2) and 3) of the lemma. The remaining cases can be verified similarly using (4.9) and (4.10). □

The boundary of Δ is the union of the boundary faces $\Delta(p_1,p_2,p_3)$, $\Delta(p_1,p_2,p_4)$, $\Delta(p_1,p_3,p_4)$ and $\Delta(p_2,p_3,p_4)$. So in order to find the equilibria on $\mathrm{bd}(\Delta)$, we can investigate each face separately. The following proposition, the proof of which follows from the convergence results and methods in [135], determines the interior equilibrium points in each of them.

Proposition 4.2. *Assume* (4.2) *holds. The interiors of the faces* $\Delta(p_1,p_2,p_4)$, $\Delta(p_1,p_3,p_4)$ *and* $\Delta(p_2,p_3,p_4)$ *do not contain an equilibrium point of the dynamics* (4.4). *If* $m=2n+1, n\geq 1$ *and* $R=\frac{T+S}{2}$, *or* $m=2n, n\geq 1$ *and* $R=\frac{nT+(n-1)S}{2n-1}$, *then the interior of the face* $\Delta(p_1,p_2,p_3)$ *contains the continuum of equilibrium points* $\mathcal{X}^{123}$, *and does not contain any other equilibrium. For all other values of* m *and the payoffs, the interior of* $\Delta(p_1,p_2,p_3)$ *does not contain an equilibrium point.*

Now we prove Proposition 4.1.

Proof of Proposition 4.1. In view of Proposition 4.2, there is no equilibrium point in the interior of any of $\Delta(p_1,p_2,p_3)$, $\Delta(p_1,p_2,p_4)$, $\Delta(p_1,p_3,p_4)$ and $\Delta(p_2,p_3,p_4)$, except for Cases 3) and 4) where $\mathcal{X}^{123}$ appears. Hence, all of the rest of the boundary equilibrium points are located on the 6 edges of the simplex. The edges $\Delta(p_1,p_2)=\mathcal{X}^{12}$ and $\Delta(p_3,p_4)=\mathcal{X}^{34}$ are always a continuum of equilibrium points. The vertices p_1,p_2,p_3,p_4 are also always equilibrium points, but they are included in $\mathcal{X}^{12}$ and $\mathcal{X}^{34}$. Hence, the rest of the equilibrium points can be determined by investigating the dynamics in the interior of the remaining four edges. We just provide the analysis for the edge $\Delta(p_1,p_4)$,

and the rest can be done similarly. For the dynamics on this edge, after using the constraint $x_1 + x_4 = 1$, we obtain

$$\dot{x}_1 = (x_1 - 1)\Big((a'_{41} + a'_{14})x_1 - a'_{14}\Big)x_1. \tag{4.11}$$

In view of Lemma 4.1, $a'_{14}, a'_{41} > 0$ regardless of the payoffs T, R, S, P. Hence, $x_1^* = \frac{a'_{14}}{a'_{14}+a'_{41}}$ belongs to $(0,1)$ and hence is an equilibrium point of the differential equation (4.11). Correspondingly, the vector $x^* = \begin{bmatrix} x_1^* & 0 & 0 & (1-x_1^*) \end{bmatrix}^\top = x^{14}$ is an equilibrium point of the dynamics (4.4). Since x_1^* is the unique interior equilibrium point of (4.11), it can be concluded that $\mathrm{int}(\Delta(p_1, p_4)) \cap \Delta^{oo} = \{x^{14}\}$. □

The local stability of the equilibrium points generally depends on the payoffs in A, and can be determined based on the convergence results in this section. However, the following result guarantees the asymptotic stability of x^{14} for all payoffs satisfying (4.2).

Proposition 4.3. *Assume* (4.2) *holds. Then* x^{14} *is asymptotically stable.*

Proof. The proof follows Proposition 4.7 and Lemma 4.6 in Appendix 4.5.1. □

Interior equilibrium point

The dynamics (4.4), may or may not possess an interior equilibrium depending on the payoff matrix A. As shown in the following proposition, if the dynamics have an interior equilibrium, it is unique and equal to

$$x^{int} = \begin{bmatrix} (a'_{42} - a'_{32})(a'_{13}a'_{24} - a'_{14}a'_{23}) \\ (a'_{31} - a'_{41})(a'_{13}a'_{24} - a'_{14}a'_{23}) \\ (a'_{24} - a'_{14})(a'_{31}a'_{42} - a'_{32}a'_{41}) \\ (a'_{13} - a'_{23})(a'_{31}a'_{42} - a'_{32}a'_{41}) \end{bmatrix} /r$$

where a'_{ij} are the entries of A' in (4.5), and

$$r = (a'_{13}a'_{24} - a'_{14}a'_{23})(a'_{31} - a'_{41} + a'_{42} - a'_{32}) + (a'_{31}a'_{42} - a'_{32}a'_{41})(a'_{13} - a'_{23} + a'_{24} - a'_{14}) > 0. \tag{4.12}$$

The positiveness of r can be derived from (4.2). Define the following constants based on the entries a'_{ij} of A':

$$b_1 = -\frac{a'_{13} - a'_{23}}{a'_{14} - a'_{24}} = \frac{\lceil \frac{m-2}{2} \rceil S + \lfloor \frac{m}{2} \rfloor T - (m-1)R}{(m-1)(S-P)},$$
$$b_2 = -\frac{a'_{42} - a'_{32}}{a'_{41} - a'_{31}} = \frac{\lceil \frac{m-2}{2} \rceil T + \lfloor \frac{m}{2} \rfloor S - (m-1)P}{(m-1)(T-R)}.$$

Proposition 4.4. *Assume* (4.2) *holds. It follows that*

1. *if* $S < R < \frac{T+S}{2}$ *or if* $m = 2n, n \geq 1$ *and* $\frac{T+S}{2} \leq R < \frac{nT+(n-1)S}{2n-1}$, *then the dynamics* (4.4) *possess exactly one interior equilibrium point* x^{int} *that is a hyperbolic saddle with two negative eigenvalues; additionally, for all initial conditions on the open line segment*

$$\mathcal{L}^{int} = \{x \in \mathrm{int}(\Delta) \,|\, x_1 = b_2 x_2, x_4 = b_1 x_3\},$$

the solution trajectory converges to x^{int}*;*

2. otherwise, the dynamics have no interior equilibrium point.

For the proof, we study the evolution of the ratios $\frac{x_1}{x_2}$ and $\frac{x_4}{x_3}$, which due to the block anti-diagonal structure of the payoff matrix A', are crucial in determining the asymptotic behavior of the replicator dynamics and are explained as follows.

Lemma 4.2. *Let $x(0) \in \text{int}(\Delta)$. Then $\frac{d}{dt}\left(\frac{x_1}{x_2}\right)$ is greater than (resp. equal to, resp. less than) 0 if and only if $\frac{x_4}{x_3}$ is greater than (resp. equal to, resp. less than) b_1. Similarly, $\frac{d}{dt}\left(\frac{x_4}{x_3}\right)$ is greater than (resp. equal to, resp. less than) 0 if and only if $\frac{x_1}{x_2}$ is greater than (resp. equal to, resp. less than) b_2.*

Proof. In view of Lemma 1.3, $x(0) \in \text{int}(\Delta)$ implies $x(t) \in \text{int}(\Delta)$ for all t. Hence, $0 < x_i(t) < 1, i = 1, \dots, 4$, for all t. So it is possible to define the ratio $\frac{x_i}{x_j}(t), i, j = 1, \dots, 4$ and calculate its time derivative using [181, Eq. 3.6] as

$$\frac{d}{dt}\Big(\frac{x_i}{x_j}\Big) = [u(p_i, x) - u(p_j, x)]\frac{x_i}{x_j}.$$

Consider the payoff matrix A' and let $i = 1, j = 2$ and $i = 3, j = 4$ to obtain the following two equations

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{x_1}{x_2}\Big) &= [\underbrace{(a'_{13} - a'_{23})}_{a'_3} x_3 + \underbrace{(a'_{14} - a'_{24})}_{a'_4} x_4]\Big(\frac{x_1}{x_2}\Big), \\
\frac{d}{dt}\Big(\frac{x_4}{x_3}\Big) &= [\underbrace{(a'_{41} - a'_{31})}_{a'_1} x_1 + \underbrace{(a'_{42} - a'_{32})}_{a'_2} x_2]\Big(\frac{x_4}{x_3}\Big).
\end{aligned} \tag{4.13}$$

In view of Lemma 4.1, $a'_1, a'_4 > 0$. Hence, because of (4.13),

$$\frac{d}{dt}\Big(\frac{x_1}{x_2}\Big) > 0 \Leftrightarrow a'_3 x_3 + a'_4 x_4 > 0 \overset{a'_4>0}{\Longleftrightarrow} \frac{x_4}{x_3} > -\frac{a'_3}{a'_4} = b_1,$$

$$\frac{d}{dt}\Big(\frac{x_4}{x_3}\Big) > 0 \Leftrightarrow a'_1 x_1 + a'_2 x_2 > 0 \overset{a'_1>0}{\Longleftrightarrow} \frac{x_1}{x_2} > -\frac{a'_2}{a'_1} = b_2.$$

This proves the " greater than" cases. The " equal to" and " less than" cases can be proven similarly. □

Determining the signs of b_1 and b_2 will prove useful, and is clarified in the following lemma.

Lemma 4.3. *It holds that $b_2 > 0$. Moreover, $b_1 > 0$ if and only if $a'_{13} < a'_{23}$ where a'_{ij} are the entries of A' in* (4.5).

Proof. In view of Lemma 4.1, $a'_{32} > a'_{42}$ and $a'_{41} > a'_{31}$. Hence, $b_2 > 0$ regardless of the payoffs in A'. Moreover, the inequality $a'_{14} > a'_{24}$ also always holds. Hence, $b_1 > 0$ if and only if $a'_{13} < a'_{23}$. □

Now we proceed to the proof of Lemma 4.2.

Proof of Proposition 4.4. Consider Case 1). In view of Lemma 4.3 and Lemma 4.1, $b_1, b_2 > 0$. Then each of the following two sets define a plane in the simplex

$$\mathcal{P}^1 = \left\{ x \in \Delta \,|\, \frac{x_4}{x_3} = b_1 \right\}, \quad \mathcal{P}^2 = \left\{ x \in \Delta \,|\, \frac{x_1}{x_2} = b_2 \right\}.$$

In view of Lemma 4.2, on each side of the plane $\mathcal{P}^1$ (resp. $\mathcal{P}^2$), the quantity $\frac{x_1}{x_2}$ (resp. $\frac{x_4}{x_3}$) either increases or decreases. Hence, if an interior equilibrium point exists, it has to lie on the interior of the intersection of the two planes $\mathcal{P}^1$ and $\mathcal{P}^2$, which is the open line segment $\mathcal{L}^{int}$. According to Lemma 4.2, $\mathcal{L}^{int}$ is invariant under the replicator dynamics (4.4). The dynamics of x_2 on $\mathcal{L}^{int}$ can be expressed as

$$\dot{x}_2 = k(fx_2 - g)(rx_2 - s)x_2 \tag{4.14}$$

where

$$\begin{aligned} k &= \frac{1}{(a'_{41} - a'_{31})^2(a'_{13} - a'_{23} + a'_{24} - a'_{14})} > 0, \\ f &= a'_{32} - a'_{42} + a'_{41} - a'_{31} > 0, \\ g &= a'_{41} - a'_{31} > 0, \\ s &= (a'_{13}a'_{24} - a'_{14}a'_{23})(a'_{31} - a'_{41}) > 0, \end{aligned}$$

and r is defined in (4.12). The equilibrium points of (4.14) are $x_2^* = 0, \frac{s}{r}, \frac{g}{f}$, which are easily proven to be unstable, stable and unstable, respectively. Hence, for any initial condition on $\mathcal{L}^{int}$, the trajectory $x(t)$ converges to $x^* \in \mathcal{L}^{int}$ where $x_2^* = \frac{s}{r}$. By using the constraints $\sum_{i=1}^4 x_i^* = 1$ and $x^* \in \mathcal{L}^{int}$, we get that $x^* = x^{int}$. Hence, x^{int} is an interior equilibrium, and for all $x(0) \in \mathcal{L}^{int}$, $x(t) \to x^{int}$. Now the eigenvalues of x^{int} are determined. Consider the replicator dynamics (4.4). Replace the vector x by $\hat{x} = \begin{bmatrix} x_1 & x_2 & x_3 & 1 - x_1 - x_2 - x_3 \end{bmatrix}^\top$, and eliminate the differential equation for $\dot{x}_4$ to get a 3rd order system. Then, the characteristic equation of the corresponding Jacobian matrix about x^{int} is $\lambda^3 + a\lambda^2 + b\lambda + c = 0$ where $a, b, c, \in \mathbb{R}$. It can be verified that $c = ab$ and that $c > 0 > a, b$. Hence, the corresponding eigenvalues of x^{int} are $a, -a, -b$, which completes the proof of this case.

Now consider Case 2) where $a'_{13} \geq a'_{23}$. Hence, $b_1 \leq 0$ in view of Lemma 4.3. Hence, $\mathcal{P}^1$ does not intersect Δ implying that the ratio $\frac{x_4}{x_3}$ is always greater than b_1. Hence, in view of Lemma 4.2, $\frac{x_1}{x_2}$ monotonically increases in $\text{int}(\Delta)$. Hence, there is no interior equilibrium point in this case. □

4.3.2 Trajectories starting on an edge

Clearly each vertex is an equilibrium point of the replicator dynamics (4.4). Hence, it remains to investigate the dynamics in the interior of each edge. Due to invariance, the convergence analysis of the dynamics (4.4) on an edge $\Delta(p_k, p_j), k, j \in \{1, 2, 3, 4\}, k \neq j$, can be reduced to the analysis of the following 2-dimensional replicator dynamics

$$\dot{x}_i = [p_i^\top \hat{A}\hat{x} - \hat{x}^\top \hat{A}\hat{x}]x_i, \qquad i = k, j$$

where

$$\hat{x} = \begin{bmatrix} x_k \\ x_j \end{bmatrix}, \quad \hat{A}_{kj} = \begin{bmatrix} a_{kk} & a_{kj} \\ a_{jk} & a_{kk} \end{bmatrix}.$$

The stability analysis of the above dynamics can be found in [181, Section 3.1.4], implying that a trajectory starting from an edge always converges to an equilibrium on that edge. In general, Δ contains 6 edges on its boundary, two of which, $\Delta(p_1, p_2)$ and $\Delta(p_3, p_4)$, are a continuum of equilibrium points; the analysis of $\Delta(p_1, p_3)$ and $\Delta(p_2, p_3)$ can be found in [135], and the other two can be analyzed according to [181].

4.3.3 Trajectories starting in the interior of a planar face

The following proposition provides a general convergence result for each planar boundary face.

Proposition 4.5. *If $x(0)$ belongs to one of the faces $\Delta(p_1, p_2, p_3)$, $\Delta(p_1, p_2, p_4)$, $\Delta(p_1, p_3, p_4)$ or $\Delta(p_2, p_3, p_4)$, then $x(t)$ converges to a point in that face as $t \to \infty$.*

A detailed convergence analysis of the face $\Delta(p_1, p_2, p_3)$ can be found in [135]. For the other three faces, the reader may refer to [24] where some general stability analyses have been performed for a planar simplex.

4.3.4 Trajectories starting in the interior of the simplex

Dynamics in the four sections made by the crucial ratios

When b_1 and b_2 are positive, the ratios $\frac{x_1}{x_2}$ and $\frac{x_4}{x_3}$ divide the simplex into the four following zones:

$$\mathcal{D}^{14} = \left\{ x \in \text{int}(\Delta) \,|\, \frac{x_1}{x_2} > b_2,\ \frac{x_4}{x_3} > b_1 \right\}, \quad \mathcal{D}^{23} = \left\{ x \in \text{int}(\Delta) \,|\, \frac{x_4}{x_3} < b_2,\ \frac{x_4}{x_3} < b_1 \right\}.$$
$$\mathcal{Y}^{14} = \left\{ x \in \text{int}(\Delta) \,|\, \frac{x_1}{x_2} > b_2,\ \frac{x_4}{x_3} < b_1 \right\}, \quad \mathcal{Y}^{23} = \left\{ x \in \text{int}(\Delta) \,|\, \frac{x_1}{x_2} < b_2,\ \frac{x_4}{x_3} > b_1 \right\}.$$

We investigate interior trajectories of the simplex by studying the dynamics in the above zones and start by $\mathcal{D}^{14}$ and $\mathcal{D}^{23}$.

Lemma 4.4. *$\mathcal{D}^{14}$ and $\mathcal{D}^{23}$ are positively invariant under* (4.4).

Proof. First $\mathcal{D}^{14}$ is shown to be positively invariant. Assume the contrary, i.e., a trajectory $x(t)$ starts from some point in $\mathcal{D}^{14}$ at $t = t^0$ but does not belong to $\mathcal{D}^{14}$ at some time $t^* > t^0$. Due to the continuity of the trajectory, there exists some time $t^1 \in (t^0, t^*)$ at which the trajectory intersects the boundary of $\mathcal{D}^{14}$. Hence, at least one of the followings happen

$$\frac{x_1}{x_2}(t^1) = b_2, \quad \frac{x_4}{x_3}(t^1) = b_1.$$

Without loss of generality, assume the first case happens. Then $\frac{x_1}{x_2}(t^1) < \frac{x_1}{x_2}(t^0)$. Hence, $\frac{d}{dt}\left(\frac{x_1}{x_2}\right)$ must be negative at some time $t^2 \in (t^0, t^1)$. Hence, due to the continuity of the time-derivative of $\frac{x_1}{x_2}$, $\frac{d}{dt}\left(\frac{x_1}{x_2}\right)$ is zero at some time $t^3 \in (t^0, t^2)$. Hence, in view of Lemma 4.2, $\frac{x_4}{x_3}(t^3) = b_1$. This implies that the trajectory has intersected the boundary of $\mathcal{D}^{14}$ at some time earlier than t^1, a contradiction. Hence, if a trajectory starts in $\mathcal{D}^{14}$ at some time $t = t^0$, it remains there afterwards. Similarly the positive invariance of $\mathcal{D}^{23}$ can be shown. □

Proposition 4.6. *Consider a trajectory $x(t)$ of the dynamics (4.4) that passes through x^0 at some time t^0. If $x^0 \in \mathcal{D}^{14}$, then one of the following cases happen*

$$\lim_{t\to\infty} x(t) = x^{14} \quad \text{or} \quad \lim_{t\to\infty} x(t) = x^* \in \mathcal{X}^{12} \cap \Delta^{NE}.$$

If $x^0 \in \mathcal{D}^{23}$, then

$$\lim_{t\to\infty} x(t) = x^* \in (\{x^{23}\} \cup \mathcal{X}^{12}) \cap \Delta^{NE}.$$

Proof. Consider the case when $x^0 \in \mathcal{D}^{14}$. In view of Lemma 4.4, $x(t) \in \mathcal{D}^{14}$ for all $t \geq t^0$. Hence, both inequalities $\frac{x_1}{x_2}(t) > b_2$ and $\frac{x_4}{x_3}(t) > b_1$ hold for all $t \geq t^0$. Hence, in view of Lemma 4.2, both ratios $\frac{x_4}{x_3}$ and $\frac{x_1}{x_2}$ monotonically increase with time. Hence, both ratios converge to either a constant or ∞. In case one of the ratios, e.g., $\frac{x_1}{x_2}$, converges to a constant, that constant must be strictly positive. This follows from the fact that $\frac{x_1}{x_2}(t^0) > 0$ and that $\frac{x_1}{x_2}$ monotonically increases. In general, one of the following cases may occur:

1) $\frac{x_1}{x_2} \to \alpha > 0$ and $\frac{x_4}{x_3} \to \beta > 0$. Hence, x converges to the following line segment

$$\mathcal{L}^{\alpha,\beta} = \{x \in \Delta \,|\, x_1 = \alpha x_2, x_4 = \beta x_3\}.$$

In view of Corollary 3.1 in Chapter 3, $x \to \mathcal{L}^{\alpha,\beta} \cap \Delta^o$. In what follows, it is shown that $\mathrm{int}(\mathcal{L}^{\alpha,\beta}) \cap \Delta^o = \emptyset$. First note that $\alpha > b_2$. This can be proven by contradiction: Assume that $\alpha \leq b_2$. Since $x(t^0) \in \mathcal{D}^{14}$, it holds that $\frac{x_1}{x_2}(t^0) > b_2$. Hence, $\frac{x_1}{x_2}(t^0) > b_2 \geq \alpha$. Then, due to the continuity of the trajectory, there exists some time $t^1 > t^0$ such that $\frac{x_1}{x_2}(t^1) = b_2$. Hence, $x(t^1) \notin \mathcal{D}^{14}$, which contradicts the invariance property of $\mathcal{D}^{14}$. So $\alpha > b_2$. Now note that $\mathrm{int}(\mathcal{L}^{\alpha,\beta}) \subseteq \mathrm{int}(\Delta)$. On the other hand, in view of Lemma 4.4, the only interior equilibrium of the system (if there exists any), belongs to the plane $\left\{x \in \Delta \,|\, \frac{x_1}{x_2} = b_2\right\}$. However, as it was discussed above, $\frac{x_1}{x_2} \to \alpha > b_2$. Hence, $\mathrm{int}(\Delta) \cap \Delta^o = \emptyset$. So $\mathrm{int}(\mathcal{L}^{\alpha,\beta}) \cap \Delta^o = \emptyset$. Thus, $x \to \mathrm{bd}(\mathcal{L}^{\alpha,\beta}) \cap \Delta^o$. The boundary of $\mathcal{L}^{\alpha,\beta}$ consists of the following two points, each of which is an equilibrium:

$$x^\alpha = \left[\tfrac{\alpha}{1+\alpha} \quad \tfrac{1}{1+\alpha} \quad 0 \quad 0\right]^\top \in \mathcal{X}^{12}, \quad x^\beta = \left[0 \quad 0 \quad \tfrac{1}{1+\beta} \quad \tfrac{\beta}{1+\beta}\right]^\top \in \mathcal{X}^{34}.$$

According to Lemma 4.7 in Appendix 4.5.2, if x converges to a point, it must belong to Δ^{NE}. However, $x^\beta \notin \Delta^{NE}$ in view of Lemma 4.8 in Appendix 4.5.2. Hence, $x \not\to x^\beta$ implying that $x \to x^\alpha$. On the other hand, $x^\alpha \in \mathcal{X}^{12}$ and x^α must belong to Δ^{NE}. Hence, $x \to x^* \in \mathcal{X}^{12} \cap \Delta^{NE}$.

2) $\frac{x_1}{x_2} \to \alpha > 0$ and $\frac{x_4}{x_3} \to \infty$. Hence, x converges to the following line segment

$$\mathcal{L}^{\alpha,\infty} = \{x \in \Delta \,|\, x_1 = \alpha x_2, x_3 = 0\}.$$

Due to Corollary 3.1, x converges to an equilibrium or a continuum of equilibria on $\mathcal{L}^{\alpha,\infty}$. On the other hand, $\mathcal{L}^{\alpha,\infty}$ lies on the face $\Delta(p_1, p_2, p_4)$, and in view of Proposition 4.2, no interior equilibrium exists on this face. Hence, x converges to the intersection of $\mathcal{L}^{\alpha,\infty}$ with the boundary of $\Delta(p_1, p_2, p_4)$ which is $\{x^\alpha, p_4\}$. However, $p_4 \notin \Delta^{NE}$ and hence $x \not\to p_4$ in view of Lemma 4.7. Hence, $x \to x^\alpha$. So, similar to the previous case, $x \to x^* \in \mathcal{X}^{12} \cap \Delta^{NE}$.

3) $\frac{x_1}{x_2} \to \infty$ and $\frac{x_4}{x_3} \to \beta > 0$. Similar to the previous case, it can be shown that $x \to x^\beta$ or $x \to p_1$. However, neither x^β nor p_1 belongs to Δ^{NE}. Hence, this case never happens.

4) $\frac{x_1}{x_2} \to \infty$ and $\frac{x_4}{x_3} \to \infty$. Hence, x converges to the following line segment

$$\mathcal{L}^{\infty,\infty} = \{x \in \Delta \,|\, x_2 = 0, x_3 = 0\} = \Delta(p_1, p_4).$$

Due to Corollary 3.1, $x \to \Delta(p_1, p_4) \cap \Delta^o = \{p_1, x^{14}, p_4\}$. On the other hand, $p_1, p_4 \notin \Delta^{NE}$. Hence, $x \to x^{14}$ in view of Lemma 4.7.

Summarizing the above four cases completes the proof for when $x^0 \in \mathcal{D}^{14}$. Now let $x^0 \in \mathcal{D}^{23}$. By following the procedure for when $x^0 \in \mathcal{D}^{14}$, it can be shown that both ratios $\frac{x_4}{x_3}$ and $\frac{x_1}{x_2}$ converge either to a positive constant or to 0. In general, one of the following cases may occur:

1*) $\frac{x_1}{x_2} \to \alpha > 0$ and $\frac{x_4}{x_3} \to \beta > 0$. Similar to when $x^0 \in \mathcal{D}^{14}$, this case results in $x \to x^* \in \mathcal{X}^{12} \cap \Delta^{NE}$.

2*) $\frac{x_1}{x_2} \to \alpha > 0$ and $\frac{x_4}{x_3} \to 0$. Hence, x converges to the following line segment

$$\mathcal{L}^{\alpha,0} = \{x \in \Delta \,|\, x_1 = \alpha x_2, x_4 = 0\}.$$

In view of Corollary 3.1, $x \to \mathcal{L}^{\alpha,0} \cap \Delta^o$. Clearly $\mathcal{L}^{\alpha,0} \subseteq \Delta(p_1, p_2, p_3)$. On the other hand, according to Proposition 4.2, $\text{int}(\Delta(p_1, p_2, p_3)) \cap \Delta^o$ either is empty or equals to $\mathcal{X}^{123}$. In view of Theorem 4.1, the second case only happens when $m = 2n + 1, n \geq 1$ and $R = \frac{T+S}{2}$, or $m = 2n, n \geq 1$ and $R = \frac{nT+(n-1)S}{2n-1}$. However, for both of these values of R, it can be verified that $b_1 < 0$. Hence, $\mathcal{D}^{23} = \emptyset$, which contradicts the assumption $x^0 \in \mathcal{D}^{23}$. Hence, $\text{int}(\Delta(p_1, p_2, p_3)) \cap \Delta^o = \emptyset$. So $\text{int}(\mathcal{L}^{\alpha,0}) \cap \Delta^o = \emptyset$ and $x \to \text{bd}(\mathcal{L}^{\alpha,0})$. Thus, $x \to \{x^\alpha, p_3\}$. However, $p_3 \notin \Delta^{NE}$ and hence $x \not\to p_3$, in view of Lemma 4.7. Hence, $x \to x^\alpha$ resulting in $x \to x^* \in \mathcal{X}^{12} \cap \Delta^{NE}$.

3*) $\frac{x_1}{x_2} \to 0$ and $\frac{x_4}{x_3} \to \beta > 0$. Hence, x converges to the following line segment

$$\mathcal{L}^{0,\beta} = \{x \in \Delta \,|\, x_1 = 0, x_4 = \beta x_3\}.$$

Similar to the previous case, it can be shown that $x \to \{x^\beta, p_2\}$. Hence, in view of Lemma 4.7, $x \to \{x^\beta, p_2\} \cap \Delta^{NE}$. So $x \to \{p_2\} \cap \Delta^{NE}$ since $x^\beta \notin \Delta^{NE}$. On the other hand, $p_2 \in \mathcal{X}^{12}$. Hence, $x \to x^* \in \mathcal{X}^{12} \cap \Delta^{NE}$.

4*) $\frac{x_1}{x_2} \to 0$ and $\frac{x_4}{x_3} \to 0$. Hence, x converges to the following line segment

$$\mathcal{L}^{0,0} = \{x \in \Delta \,|\, x_1 = 0, x_4 = 0\} = \Delta(p_2, p_3).$$

Due to Corollary 3.1, $x \to \Delta(p_2, p_3) \cap \Delta^o = \{p_2, x^{23}, p_3\}$. On the other hand, $p_3 \notin \Delta^{NE}$. Hence, $x \to \{x^{23}, p_2\} \cap \Delta^{NE}$ in view of Lemma 4.7. Since $p_2 \in \mathcal{X}^{12}$, it can be concluded that $x \to x^* \in (\mathcal{X}^{12} \cup \{x^{23}\}) \cap \Delta^{NE}$.

By summarizing the above cases, the proof for when $x^0 \in \mathcal{D}^{23}$ is complete. □

Lemma 4.5. *Consider a trajectory $x(t)$ of the dynamics (4.4) that passes through x^0 at some time t^0. If $x^0 \in \mathcal{Y}^{14}$, then either $x(t)$ leaves $\mathcal{Y}^{14}$ after some finite time, or*

$$\lim_{t\to\infty} x(t) = x^{int} \quad or \quad \lim_{t\to\infty} x(t) = x^* \in \mathcal{X}^{12} \cap \Delta^{NE}.$$

If $x^0 \in \mathcal{Y}^{23}$, then either $x(t)$ leaves $\mathcal{Y}^{23}$ after some finite time, or

$$\lim_{t\to\infty} x(t) = x^{int} \quad or \quad \lim_{t\to\infty} x(t) = x^* \in (\mathcal{X}^{12} \cup \mathcal{X}^{123}) \cap \Delta^{NE}.$$

Proof. Consider the case when $x^0 \in \mathcal{Y}^{14}$. If x leaves $\mathcal{Y}^{14}$ after some finite time, the conclusion can be drawn directly. So let $\mathcal{Y}^{14}$ be invariant. Then the inequalities in the definition of $\mathcal{Y}^{14}$ hold for all $t \geq t^0$. Hence, in view of Lemma 4.2, $\frac{x_1}{x_2}$ monotonically decreases and hence converges to a constant $\alpha \geq b_2$, and $\frac{x_4}{x_3}$ monotonically increases and hence converges to a constant $\beta \leq b_1$ as $t \to \infty$. Hence, $x(t)$ converges to the line segment $\mathcal{L}^{\alpha,\beta} = \{x \in \Delta \,|\, x_1 = \alpha x_2, x_4 = \beta x_3\}$. So based on Corollary 3.1, $x(t)$ converges to $\mathcal{L}^{\alpha,\beta} \cap \Delta^o$. On the other hand, Δ^o includes at most one interior equilibrium point x^{int} according to Proposition 4.4. Hence, either $x(t) \to x^{int}$ or $x(t) \to \mathcal{L}^{\alpha,\beta} \cap \Delta^{oo}$. The first case leads to the conclusion directly, so consider the second case. First note that $\alpha > 0$ since $b_2 > 0$ in view of Lemmas 4.3 and 4.1. Moreover, $\beta > 0$ since $\frac{x_4}{x_3}$ monotonically increases from $\frac{x_4}{x_3}(0) > 0$ to β. Hence, $\alpha, \beta > 0$. So on the set $\mathcal{L}^{\alpha,\beta} \cap \mathrm{bd}(\Delta)$, either $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$ holds. Then $\mathcal{L}^{\alpha,\beta} \cap \mathrm{bd}(\Delta)$ equals a point $x^* \in \mathcal{X}^{12} \cup \mathcal{X}^{34}$. On the other hand, $\Delta^{oo} \subseteq \mathrm{bd}(\Delta)$. Hence, since $\mathcal{X}^{12} \cup \mathcal{X}^{34} \subseteq \Delta^{oo}$ it holds that $\mathcal{L}^{\alpha,\beta} \cap \Delta^{oo} = x^* \in \mathcal{X}^{12} \cup \mathcal{X}^{34}$. Thus, in view of Lemma 4.7 in Appendix 4.5.2, $x(t) \to x^* \in (\mathcal{X}^{12} \cup \mathcal{X}^{34}) \cap \Delta^{NE}$. On the other hand, $\mathcal{X}^{34} \cap \Delta^{NE} = \emptyset$ according to Lemma 4.8 in Appendix 4.5.2. Hence, $x(t) \to x^* \in \mathcal{X}^{12} \cap \Delta^{NE}$, which completes the proof of this part.

Now consider the case when $x^0 \in \mathcal{Y}^{23}$ and $\mathcal{Y}^{23}$ is invariant (otherwise, the result is trivial). Hence, in view of Lemma 4.2, $\frac{x_1}{x_2}$ monotonically increases and hence converges to a constant $\alpha \leq b_2$, and $\frac{x_4}{x_3}$ monotonically decreases and hence converges to a constant $\beta \geq b_1$ as $t \to \infty$. So similar to the previous case, either $x(t) \to x^{int}$ or $x(t) \to \mathcal{L}^{\alpha,\beta} \cap \Delta^{oo}$. Again the first case leads to the conclusion directly, so consider the second. It must be true that $\alpha > 0$ since $\frac{x_1}{x_2}$ monotonically increases from $\frac{x_1}{x_2}(0) > 0$ to α. If β is also positive, then the same as when $x^0 \in \mathcal{Y}^{14}$ takes place, which makes the result trivial. So let $\beta = 0$. Then $\mathcal{L}^{\alpha,\beta} = \{x \in \Delta \,|\, x_1 = \alpha x_2, x_4 = 0\}$. Hence, in view of Theorem 4.1, $\mathcal{L}^{\alpha,\beta} \cap \Delta^{oo} = x^* \in \{x^{13}\} \cup \mathcal{X}^{12} \cup \mathcal{X}^{123}$. So in view of Lemma 4.7 and 4.9 in Appendix 4.5.2, $x(t) \to x^* \in (\mathcal{X}^{12} \cup \mathcal{X}^{123}) \cap \Delta^{NE}$, which completes the proof. □

Global results

We proceed to the global convergence analysis. As one would expect, the convergence results depend on the payoffs and to some extent also on m. We provide the results from small to large R via the following four theorems.

Theorem 4.1. *Assume* (4.2) *holds. Let* $x(0) \in \mathrm{int}(\Delta)$. *Denote the 2-dimensional stable manifold of* x^{int} *by* $W^s(x^{int})$. *If* $S < R < \frac{T+S}{2}$, *then*

1. $x(0) \in W^s(x^{int}) \Rightarrow \lim_{t\to\infty} x(t) = x^{int}$;
2. $x(0) \notin W^s(x^{int}) \Rightarrow \lim_{t\to\infty} x(t) = x^{14}$ *or* x^{23};
3. x^{14} *and* x^{23} *are asymptotically stable and their basins of attraction are separated by* $W^s(x^{int})$;
4. $x(0) \in \mathcal{D}^{14} \Rightarrow \lim_{t\to\infty} x(t) = x^{14}$;
5. $x(0) \in \mathcal{D}^{23} \Rightarrow \lim_{t\to\infty} x(t) = x^{23}$.

Proof. Case 1) of the theorem is a direct result of Theorem 4.4. Now we proceed to Case 2). According to Lemma 4.3 and Lemma 4.1, $b_1, b_2 > 0$. Hence, the interior of the simplex can be written as

$$\text{int}(\Delta) = \mathcal{D}^{14} \cup \mathcal{D}^{23} \cup \mathcal{Y}^{14} \cup \mathcal{Y}^{23} \cup \mathcal{L}^{int} \cup \hat{\mathcal{P}}_{11} \cup \hat{\mathcal{P}}_{12} \cup \hat{\mathcal{P}}_{21} \cup \hat{\mathcal{P}}_{22} \tag{4.15}$$

where $\mathcal{L}^{int}$ is defined in Theorem 4.4 and

$$\begin{aligned} \hat{\mathcal{P}}_{11} &= \{x \in \text{int}(\Delta) \,|\, \frac{x_4}{x_3} = b_1, \frac{x_1}{x_2} > b_2\}, \ \hat{\mathcal{P}}_{12} = \{x \in \text{int}(\Delta) \,|\, \frac{x_4}{x_3} = b_1, \frac{x_1}{x_2} < b_2\}, \\ \hat{\mathcal{P}}_{21} &= \{x \in \text{int}(\Delta) \,|\, \frac{x_1}{x_2} = b_2, \frac{x_4}{x_3} > b_1\}, \ \hat{\mathcal{P}}_{22} = \{x \in \text{int}(\Delta) \,|\, \frac{x_1}{x_2} = b_2, \frac{x_4}{x_3} < b_1\}. \end{aligned} \tag{4.16}$$

Hence, $x(0)$ belongs to one of the sets on the right hand side of (4.15). If $x(0) \in \mathcal{D}^{14}$, then in view of Proposition 4.6, $x(t)$ converges to either x^{14} or a point in $\mathcal{X}^{12} \cap \Delta^{NE}$. However, in view of Lemma 4.8 in Appendix 4.5.2, $\mathcal{X}^{12} \cap \Delta^{NE} = \emptyset$. Hence, $x(t) \to x^{14}$. This proves Case 4). Similarly Case 5) can be shown. Now consider the case when $x(0) \in \mathcal{Y}^{14}$. In view of Lemma 4.5, if $x(t)$ remains in $\mathcal{Y}^{14}$, it converges to a point in $\mathcal{X}^{12} \cap \Delta^{NE}$. However, in view of Lemma 4.8, $\mathcal{X}^{12} \cap \Delta^{NE} = \emptyset$, which implies $x(t)$ leaves $\mathcal{Y}^{14}$ after some finite time. Hence, $x(t)$ enters one of the sets $\mathcal{L}^{int}$, $\hat{\mathcal{P}}_{11}$, or $\hat{\mathcal{P}}_{22}$ at some time $t^1 > 0$. If $x(t^1) \in \mathcal{L}^{int}$, then $x(t) \to x^{int}$ in view of Proposition 4.4. If $x(t^1) \in \hat{\mathcal{P}}_{11}$, then $x(t)$ enters $\mathcal{D}^{14}$ after $t = t^1$ since $\frac{x_1}{x_2} > b_2$ in $\hat{\mathcal{P}}_{11}$ and hence in view of Lemma 4.2, $\frac{x_4}{x_3}$ increases at $\hat{\mathcal{P}}_{11}$. So $x(t) \to x^{14}$ in view of Case 4). Similarly, it can be shown that if $x(t^1) \in \hat{\mathcal{P}}_{22}$, then $x(t) \to x^{23}$. Hence, if $x(0) \in \mathcal{Y}^{14}$, then $x(t)$ converges to one of the points x^{14}, x^{23} or x^{int}. The same can be shown for when $x(0) \in \mathcal{Y}^{23}$ since $\mathcal{X}^{123} \not\subseteq \Delta^{oo}$ when $R < \frac{T+S}{2}$. Moreover, the cases when $x(0)$ belongs to one of the sets $\mathcal{L}^{int}, \hat{\mathcal{P}}_{11}, \hat{\mathcal{P}}_{12}, \hat{\mathcal{P}}_{21}$ or $\hat{\mathcal{P}}_{22}$ are already included in the arguments for $\mathcal{Y}^{14}$ and $\mathcal{Y}^{23}$. Hence, $x(t)$ converges to one of x^{14}, x^{23} or x^{int}. On the other hand, only for $x(0) \in W^s(x^{int})$, $x(t) \to x^{int}$. Hence, Case 2) is proven.

Both x^{14} and x^{23} are asymptotically stable in view of Proposition 4.7 and lemma 4.6. Denote their corresponding basin of attractions by $\mathcal{B}^{14}$ and $\mathcal{B}^{23}$. Clearly $\mathcal{B}^{14}$ and $\mathcal{B}^{23}$ are disjoint. Define $\hat{\mathcal{B}}^{14} := \text{bd}(\mathcal{B}^{14}) \cap \text{int}(\Delta)$ and $\hat{\mathcal{B}}^{23} := \text{bd}(\mathcal{B}^{23}) \cap \text{int}(\Delta)$. Consider a point $x^* \in \hat{\mathcal{B}}^{14}$. The solution $x(t)$ with the initial condition x^*, converges to one of x^{14}, x^{23} or x^{int} as it was shown above. However, $x(t) \not\to x^{14}$ since $x^* \notin \mathcal{B}^{14}$. Moreover, $x^* \notin \mathcal{B}^{23}$ since $x^* \in \text{bd}(\mathcal{B}^{14})$ and $\mathcal{B}^{14} \cap \mathcal{B}^{23} = \emptyset$ and $\mathcal{B}^{23}$ is open. Hence, $x(t) \to x^{int}$. So x^{int} lies on $\mathcal{B}^{14}$. The same can be shown for $\mathcal{B}^{23}$. Now both $\hat{\mathcal{B}}^{14}$ and $\hat{\mathcal{B}}^{23}$ are 2-dimensional invariant manifolds, and for any initial condition located on them, $x(t) \to x^{int}$. On the other hand, x^{int} is hyperbolic in view of Theorem 4.4, and hence $W^s(x^{int})$ is the unique 2-dimensional invariant manifold passing through x^{int}. Hence, $\hat{\mathcal{B}}^{14}$ and $\hat{\mathcal{B}}^{23}$ coincide and are equivalent to $W^s(x^{int})$. This proves Case 3) and hence the whole. □

An example of the two-dimensional stable manifold mentioned in Theorem 4.1 is shown in Figure 4.1. For intermediate values of R, the convergence results depend on whether m is odd or even. Therefore, two separate theorems are dedicated to these values.

Theorem 4.2. *Assume* (4.2) *holds. Let* $x(0) \in \text{int}(\Delta)$. *Assume* $m = 2n, n \geq 1$. *It follows that*

1. *if* $R = \frac{T+S}{2}$, *then*

$$\lim_{t \to \infty} x(t) = x^* \in \{x^{14}, x^{int}, p_2\};$$

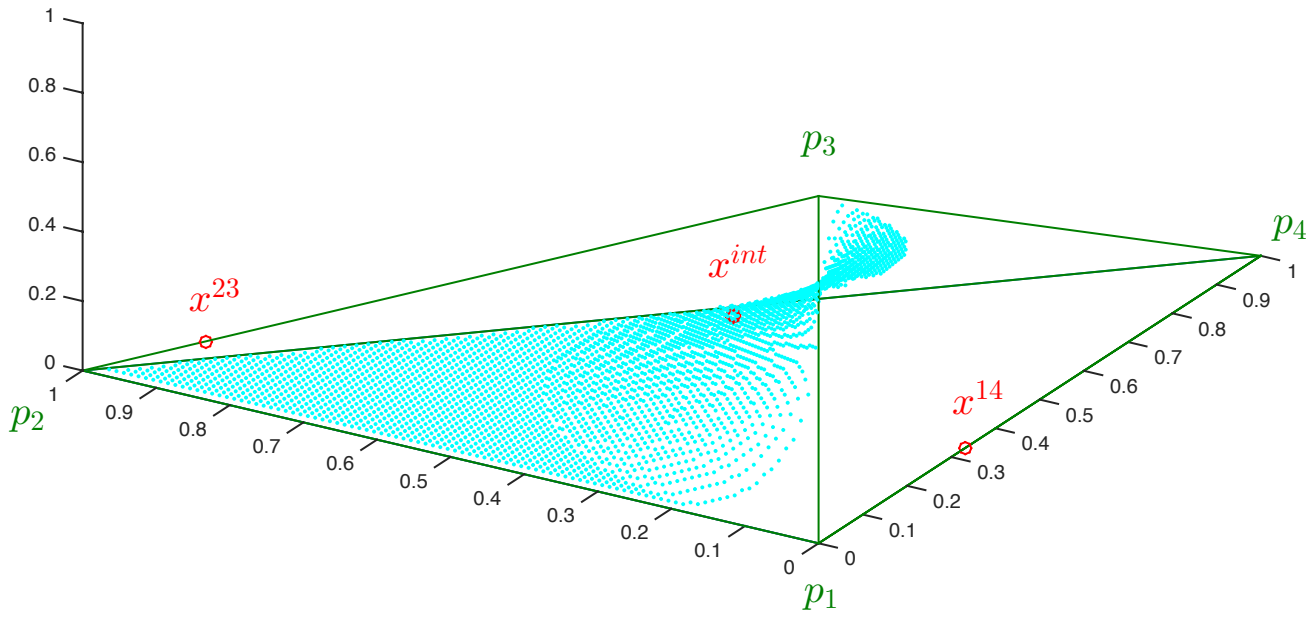


Figure 4.1: An example of the two-dimensional stable manifold mentioned in Theorem 4.1 for payoff values $T = 6$, $R = 4$, $S = 3$, $P = 2$ and the number of repetitions $m = 8$. The blue points are samples of the stable manifold.

2. *if* $\frac{T+S}{2} < R < \frac{nT+(n-1)S}{2n-1}$, *then*

$$\lim_{t\to\infty} x(t) = x^* \in \{x^{14}, x^{int}\} \cup (\mathcal{X}^{12} \cap \Delta^{NE});$$

3. *if* $R = \frac{nT+(n-1)S}{2n-1}$, *then*

$$\lim_{t\to\infty} x(t) = x^* \in \{x^{14}\} \cup \mathcal{X}^{123} \cup (\mathcal{X}^{12} \cap \Delta^{NE}).$$

Proof. In view of Lemma 4.3, $b_1, b_2 > 0$ in Cases 1) and 2). Hence, by following the same steps as in the proof of Theorem 4.1, it can be shown that $x(t) \to x^* \in \{x^{14}, x^{23}, x^{int}\} \cup (\mathcal{X}^{12} \cap \Delta^{NE})$. However, $x^{23} \notin \Delta^o$ in view of Theorem 4.1 and hence $x(t) \not\to x^{23}$. Then in view of Lemma 4.7, Case 2) is proven. Moreover, the fact that $\mathcal{X}^{12} \cap \Delta^{NE} = \{p_2\}$ for $R = \frac{T+S}{2}$, proves Case 1). For Case 3), $b_2 > 0$, but $b_1 = 0$ in view of Lemma 4.3. Hence, $\text{int}(\Delta)$ can be written as follows

$$\text{int}(\Delta) = \mathcal{D}^{14} \cup \mathcal{Y}^{14} \cup \hat{\mathcal{P}}_{11} \tag{4.17}$$

where $\hat{\mathcal{P}}_{11}$ is defined in (4.16). Then similar to the proof of Theorem 4.1, we arrive at the conclusion. □

Theorem 4.3. *Assume* (4.2) *holds. Let* $x(0) \in \text{int}(\Delta)$. *Assume* $m = 2n + 1, n \geq 1$. *It follows that*

1. *if* $R = \frac{T+S}{2}$, *then*

$$\lim_{t\to\infty} x(t) = x^* \in \{x^{14}, x^{23}\} \cup \mathcal{X}^{123};$$

2. *if* $\frac{T+S}{2} < R \leq \frac{(n+1)T+nS}{2n+1}$, *then*

$$\lim_{t\to\infty} x(t) = x^{14};$$

Proof. In view of Lemma 4.3, $b_2 > 0 \geq b_1$ in all cases. Hence, by following the same steps as in the proof of Case 3) of Theorem 4.2, it can be shown that $x(t) \to x^* \in \{x^{14}, x^{23}\} \cap (\mathcal{X}^{12} \cap \Delta^{NE}) \cap \mathcal{X}^{123}$ where $\mathcal{X}^{123}$ shows up only in Case 1) according to Theorem 4.1. Then according to Lemma 4.8 in Appendix 4.5.2, $\mathcal{X}^{12} \cap \Delta^{NE} = \emptyset$, which proves Case 1) and Case 2) except when R equals $\frac{(n+1)T+nS}{2n+1}$. When the equality happens, $\mathcal{X}^{12} \cap \Delta^{NE} = \{p_2\}$ in view of Lemma 4.9 in Appendix 4.5.2. However, in view of Lemma 4.2, $b_1 \leq 0$ implies that $\frac{x_1}{x_2}$ monotonically increases. Hence, $\frac{x_1}{x_2}(t) > \frac{x_1}{x_2}(0)$ for all $t > 0$. On the other hand, $\frac{x_1}{x_2}(0) > p_2$ since $x(0) \in \text{int}(\Delta)$. Hence, $x(t) \not\to p_2$, which completes the proof. □

Theorem 4.4. *Assume* (4.2) *holds. Let* $x(0) \in \text{int}(\Delta)$. *If* $\max\left\{\frac{\lceil\frac{m-2}{2}\rceil S+\lfloor\frac{m}{2}\rfloor T}{m-1}, \frac{\lceil\frac{m}{2}\rceil T+\lfloor\frac{m}{2}\rfloor S}{m}\right\} < R < T$, *then*

$$\lim_{t\to\infty} x(t) = x^* \in \{x^{14}\} \cup (\mathcal{X}^{12} \cap \Delta^{NE}).$$

Proof. The proof is similar to that of Case 3) in Theorem 4.3. □

The integration of the convergence results when the initial condition is in the interior of the simplex and when it is on the boundary of the simplex, yields the following corollary.

Corollary 4.1. *Assume* (4.2) *holds. For any initial condition* $x(0) \in \Delta$, *the solution* $x(t)$ *of the replicator dynamics* (4.4), *converges to a point in* Δ *as time goes to infinity.*

Therefore, no limit cycle or strange attractor can take place in the dynamics, and we always have convergence to a point.

4.3.5 Discussion

Now that we know the asymptotic behavior of the replicator dynamics (4.4) for all range of payoffs, we can proceed to the interpretation of the results in terms of the individuals playing the four types of strategies. Assume that the payoffs satisfy the snowdrift constraint (4.2). Consider a population where the portions of individuals playing $ALLC$, TFT, $STFT$ and $ALLD$ are all nonzero.

For small values of R, i.e., less than the average of T and S, almost always the population converges to one of the following states: (1) x^{14} that is a mixed population of $ALLC$ and $ALLD$ players or (2) x^{23} that is a mixed population of TFT and $STFT$ players. Both states are evolutionary (and hence asymptotically) stable (see Appendix-B). Therefore, evolutionary forces select against any mutant population at these two states. Moreover, for an zero-measure set of initial states, the population converges to x^{int} where all four types of players are present. Clearly, x^{int} is not stable and small perturbations can lead the population to one of x^{14} and x^{23}.

Now if the base game is repeated for even number of times, as R increase, the state x^{23} moves towards p_2 where only TFT players are present. When R equals the average of T and S, x^{23} coincides with p_2, and hence $STFT$ players stand out (except for those zero-measure initial conditions that lead to x^{int}). As R further increases, the single equilibrium state p_2 is expanded to the set $\mathcal{X}^{12} \cap \Delta^{NE}$. Therefore, the population either converges to x^{14} where $ALLC$ or $ALLD$

players coexist or to a state where $ALLC$ and TFT players coexist. At the same time, x^{int} is converging to the face $\Delta(p_1, p_2, p_3)$, and when R equals $\frac{nT+(n-1)S}{2n-1}$, x^{int} lies on $\mathcal{X}^{123}$ where $ALLC$, TFT and $STFT$ players coexist.

If the base game is repeated for odd number of times, $STFT$ players survive for a greater range of R. This time for R being equal to $\frac{T+S}{2}$, x^{int} lies on $\mathcal{X}^{123}$. Then suddenly, by a small increment in R, the set $\mathcal{X}^{123}$ disappears, and no population converges to x^{23}. Therefore, starting from any initial condition, the population converges to the polymorphic population of $ALLC$ and $ALLD$ players, i.e., x^{14}. This is the only situation where both conditional strategies TFT and $STFT$ are wiped out of the population, and is continued up to when R equals $\frac{(n+1)T+nS}{2n+1}$.

When R further increases, the behavior of the system is almost the same for both odd and even m. The population either converges to x^{14} where $ALLC$ and $ALLD$ coexist or to a mixed population of $ALLC$ and TFT players. Moreover, as R increases, x^{14} gets closer to p_1 where all individuals are $ALLC$ players.

In general, perhaps $STFT$ can be considered as the worst strategy in terms of survival especially for $R > \frac{T+S}{2}$. Conversely, regardless of the payoffs, there always exists a set of initial conditions for which $ALLC$ players show up in the long run. Moreover, in addition to x^{14}, all the limit states in $\mathcal{X}^{12} \cap \Delta^{NE}$ (except for p_2) have a nonzero portion of $ALLC$ players. This makes the simple $ALLC$ strategy perhaps the most robust in terms of survival and appearance in the long run.

Interestingly, x^{14} is always an evolutionary (and asymptotically) stable state of the system, regardless of the payoffs. This state consists of $\frac{S-P}{S-P+T-R}$ $ALLC$ players that can be considered as cooperators and $\frac{T-R}{S-P+T-R}$ $ALLD$ players that can be considered as defectors. On the other hand, the unique evolutionary stable state of the base game consists of $\frac{S-P}{S-P+T-R}$ C players, i.e., cooperators, and $\frac{T-R}{S-P+T-R}$ D players, i.e., defectors. Thus, the repetition of the base game and the introduction of the two conditional strategies TFT and $STFT$, does not eliminate or even change this evolutionary stable mixture of cooperators and defectors, but adds some new more-cooperative final states such as those on $\mathcal{X}^{12}$.

An important performance measure of a given population state x is the average number of times cooperation is played in the population which we call the *average cooperation level* of that population and denote it by x_C:

$$x_C := \sum_{i,j \in \{1,\dots,4\}} x_i x_j \frac{C_{ij}}{2m}$$

where C_{ij} is the number of times cooperation is played in the m rounds when two individuals playing the strategies corresponding to indices i and j are matched to play the repeated game G^m. The average cooperation level at x^{14} is $\frac{S-P}{S-P+T-R}$ since only $ALLC$ players cooperate, and reaches 1 at any state in $\mathcal{X}^{12}$ since both $ALLC$ and TFT players cooperate. Therefore, based on the convergence results, adding enough TFT players to a population of $ALLC$ and $ALLD$ players can dramatically increase the average level of cooperation, especially for large values of R, resulting in a more cooperative population. The claim does not change when $STFT$ players are also present in the population. The convergence analysis also reveals how the average cooperation level changes as R increases. Particularly, in the presence of the four types of players, increments in R make the final population more probable to become completely cooperative.

4.4 Concluding remarks

The evolution of a well-mixed population of individuals playing a repeated snowdrift game with four typical conditional strategies -ALLC, ALLD, TFT and STFT - is studied under the replicator dynamics. The asymptotic behavior of the dynamics as well as their evolutionary game-theoretical implications are demonstrated. The convergence analysis sheds light on why selfish individuals cooperate in snowdrift social dilemmas, and provides a parametric framework to control the final population portions of the four individual types. [1]

4.5 Appendix

4.5.1 Evolutionary stability: proof of Proposition 4.3

A state $x \in \Delta$ is said to be an *evolutionary stable state (strategy) (ESS) of* A if it satisfies the following two conditions [149, pp. 81]:

$$x^\top Ax \geq y^\top Ax \quad \forall y \in \Delta, \tag{4.18}$$

$$[x^\top Ax = y^\top Ax \quad \text{and} \quad y \neq x] \Rightarrow x^\top Ay > y^\top Ay. \tag{4.19}$$

The set of all evolutionary stable states is denoted by Δ^{ESS}.

Lemma 4.6 (Proposition 3.10 in [181])**.** *Every* $x \in \Delta^{ESS}$ *is asymptotically stable under the replicator dynamics* (4.4)*.*

Proposition 4.7. $x^{14} \in \Delta^{ESS}$*. Moreover,* $x^{23} \in \Delta^{ESS}$ *if* $R < \frac{T+S}{2}$*.*

Proof. The result for x^{14} is proven in the following, and that for x^{23} can be done similarly. Consider

$$Ax^{14} = \frac{1}{a'_{14} + a'_{41}} \begin{pmatrix} a'_{14}a'_{41} & a'_{24}a'_{41} & a'_{14}a'_{31} & a'_{14}a'_{41} \end{pmatrix}^\top.$$

In view of Lemma 4.1, $a'_{41} > a'_{31} \geq 0$ and $a'_{14} > a'_{24} \geq 0$. Hence, $a'_{14}a'_{41} > a'_{24}a'_{41}, a'_{14}a'_{41}$, implying that the maximum element of Ax^{14} is $a'_{14}a'_{41}$. Hence, any $y \in \Delta$ satisfying $y_2, y_3 = 0$ maximizes $y^\top Ax^{14}$. So ${x^{14}}^\top Ax^{14}$ is the maximum of $y^\top Ax^{14}$, which implies that (4.18) is in force. On the other hand, if for some $y \in \Delta$, ${x^{14}}^\top Ax^{14} = y^\top Ax^{14}$, then y maximizes $y^\top Ax^{14}$. Such a y satisfies $y_2, y_3 = 0$, which results in

$$y^\top Ax^{14} = \frac{{a'_{14}}^2 y_4 + {a'_{41}}^2 y_1}{a'_{14} + a'_{41}}, \quad y^\top Ax^{14} = (a'_{14} + a'_{41})y_1 y_4. \tag{4.20}$$

On the other hand,

$$[y_4(a'_{14} + a'_{41}) - a'_{41}]^2 \geq 0 \iff {a'_{14}}^2 y_4 + {a'_{41}}^2(1 - y_4) \geq (a'_{14} + a'_{41})^2 y_4(1 - y_4)$$

$$\iff {a'_{14}}^2 y_4 + {a'_{41}}^2 y_1 \geq (a'_{14} + a'_{41})^2 y_1 y_4 \iff \frac{{a'_{14}}^2 y_4 + {a'_{41}}^2 y_1}{a'_{14} + a'_{41}} \geq (a'_{14} + a'_{41})y_1 y_4.$$

[1] We would like to thank Dr. Hildeberto Jardón-Kojakhmetov for his technical discussions.

Hence, in view of (4.20), ${x^{14}}^\top Ay \geq y^\top Ay$. However, the equality holds only when

$$[y_4(a'_{14}+a'_{41})-a'_{41}]^2=0 \Rightarrow y_4=\frac{a'_{41}}{a'_{14}+a'_{41}}, y_1=\frac{a'_{14}}{a'_{14}+a'_{41}} \Rightarrow y=x^{14}.$$

Hence, ${x^{14}}^\top Ax^{14} > y^\top Ax^{14}$ for all $y \neq x^{14}$. So (4.19) is true, implying $x^{14} \in \Delta^{ESS}$. □

4.5.2 Nash equilibria and their relation to convergence points

Call a trajectory $x(t)$ an *interior trajectory*, if $x(0) \in \text{int}(\Delta)$. When investigating the final state (convergence point) of an interior trajectory, several equilibrium points often show up as possible candidates. In what follows, a known game theoretical result is reviewed to confine the possible candidates. Define Δ^{NE}, the subset of strategies (states) that are in Nash equilibrium with themselves [181, Section 1.5.2], by

$$\Delta^{NE}=\left\{x\in\Delta \,|\, x^\top Ax \geq y^\top Ax \quad \forall y\in\Delta\right\}.$$

Lemma 4.7. ([181, Proposition 3.5]) *If an interior trajectory $x(t)$ converges to a point x^*, then $x^* \in \Delta^{NE}$.*

Similar to Lemma 1.3, it can be easily verified that Δ^{NE} is invariant under the addition of a constant to all of the entries of a column of the payoff matrix A. Hence, we change A in the definition of Δ^{NE} with the more simple-structure payoff matrix A' in future derivations. The following lemma reveals those points of $\mathcal{X}^{12}$ and $\mathcal{X}^{34}$ that belong to Δ^{NE}.

Lemma 4.8. *Assume* (4.2) *holds. Then $\mathcal{X}^{34} \cap \Delta^{NE} = \emptyset$. Moreover,*

- *if $S < R < \frac{T+S}{2}$ or $m=2n+1, n\geq 1$ and $\frac{T+S}{2} \leq R < \frac{(n+1)T+nS}{2n+1}$, then*

$$\mathcal{X}^{12} \cap \Delta^{NE} = \emptyset;$$

- *if $m=2n+1, n\geq 1$ and $R=\frac{(n+1)T+nS}{2n+1}$, or $m=2n, n\geq 1$ and $R=\frac{T+S}{2}$, then*

$$\mathcal{X}^{12} \cap \Delta^{NE} = \{p_2\};$$

- *if $m=2n+1, n\geq 1$ and $\frac{(n+1)T+nS}{2n+1} < R < T$, or $m=2n, n\geq 1$ and $\frac{T+S}{2} < R < T$, then*

$$\mathcal{X}^{12} \cap \Delta^{NE} = \left\{\alpha p_1+(1-\alpha)p_2 \,\middle|\, \alpha \in \left[0, \min\left\{\frac{mR-\lceil\frac{m}{2}\rceil T-\lfloor\frac{m}{2}\rfloor S}{T-R}, \frac{mR-T-(m-1)P}{m(T-R)}, 1\right\}\right]\right\}.$$

Proof. Let $x \in \mathcal{X}^{34}$. Then $A'x = \begin{bmatrix} a'_{13}x_3+a'_{14}x_4 & a'_{23}x_3+a'_{24}x_4 & 0 & 0\end{bmatrix}^\top$. So based on the definition of Δ^{NE},

$$x\in\Delta^{NE} \iff a'_{13}x_3+a'_{14}x_4 \leq 0 \text{ and } a'_{23}x_3+a'_{24}x_4 \leq 0.$$

However, in view of Lemma 4.1, $a'_{13}, a'_{14}, a'_{23}, a'_{24} > 0$. Hence, because of $x_3 + x_4 = 1$ and $x_3, x_4 \geq 0$, it can be concluded that $x \notin \Delta^{NE}$. Now let $x \in \mathcal{X}^{12}$. Then

$$A'x = \begin{bmatrix} 0 & 0 & a'_{31}x_1 + a'_{32}x_2 & a'_{41}x_1 + a'_{42}x_2 \end{bmatrix}^\top.$$

Then based on the definition of Δ^{ESS}, we have

$$x \in \Delta^{NE} \iff a'_{31}x_1 + a'_{32}x_2 \leq 0 \text{ and } a'_{41}x_1 + a'_{42}x_2 \leq 0.$$

Moreover, $a'_{41}, a'_{31} > 0$ in view of Lemma 4.1. So

$$\begin{aligned} x \in \Delta^{NE} &\iff x_1 + \frac{a'_{32}}{a'_{31}}x_2 \leq 0 \text{ and } x_1 + \frac{a'_{42}}{a'_{41}}x_2 \leq 0 \\ &\iff 0 \leq x_1 \leq \min\left\{-\frac{a'_{32}}{a'_{31}}, -\frac{a'_{42}}{a'_{41}}\right\}, x_1 \leq 1. \end{aligned}$$

Hence, if $\min\left\{-\frac{a'_{32}}{a'_{31}}, -\frac{a'_{42}}{a'_{41}}\right\} < 0$, then $x \notin \Delta^{NE}$. Otherwise, $x = \alpha p_1 + (1-\alpha)p_2$ where $\alpha \in \left[0, \min\left\{-\frac{a'_{32}}{a'_{31}}, -\frac{a'_{42}}{a'_{41}}, 1\right\}\right]$. Substituting the values of a'_{ij} from A' in the above equation completes the proof. □

The following lemma reveals those singleton boundary equilibria that belong to Δ^{NE}.

Lemma 4.9. *$x^{13}, x^{24} \notin \Delta^{NE}$ and $x^{14} \in \Delta^{NE}$. Moreover, if $S < R < \frac{T+S}{2}$, or $m = 2n+1, n \geq 1$ and $R = \frac{T+S}{2}$, then $x^{23} \in \Delta^{NE}$. Otherwise, $x^{23} \notin \Delta^{NE}$.*

Proof. The sign-structure of $A'x^{13}$ is of the form $\begin{bmatrix} + & + & + & ++ \end{bmatrix}^\top$. Hence, $p_4^\top A'x^{13} > x^{13\top} A'x^{13}$. Hence, $x^{13} \notin \Delta^{NE}$ by definition. Similarly $x^{24} \notin \Delta^{NE}$ can be shown. Now the result for x^{23} is proven and that for x^{14} can be done similarly. Define $z := A'x^{23} = \begin{bmatrix} a'_{13}x_3 & a'_{23}x_3 & a'_{32}x_2 & a'_{42}x_2 \end{bmatrix}^\top$. Let $S < R < \frac{T+S}{2}$ or $m = 2n+1, n \geq 1$ and $R = \frac{T+S}{2}$. In view of Lemma 4.1, $a'_{32} > a'_{42}$ and hence $z_3 > z_4$. Similarly, $z_2 \geq z_1$. Moreover, it can be verified that $z_2 = z_3$. Hence, $z_2, z_3 = \max_{i\in\{1,\dots,4\}} z_i$. Hence, any $x \in \Delta(p_2, p_3)$, maximizes $x^\top z = x^\top A'x^{23}$ over Δ. Hence, since $x^{23} \in \Delta(p_2, p_3)$, it holds that $x^{23\top} A'x^{23} \geq y^\top A'x^{23}$ for all $y \in \Delta$. Hence, $x^{23} \in \Delta^{NE}$. For all other payoffs, either $x^{23} \notin \Delta$ or $z_1 > z_2$. The first case clearly implies $x^{23} \notin \Delta^{NE}$. For the second case, $p_1^\top A'x^{23} > x^{23\top} A'x^{23}$, which rules out x^{23} from Δ^{NE}. □

Chapter 5

Homophily, heterophily, and the diversity of messages in cheap-talk games

A simple yet seemingly effective use of preplay communication or cheap talk in evolutionary games is when players choose strategies based on whether an opponent sends the same message as they do. In addition to pure cooperation and defection, players have two new strategies in this setting: homophilic (resp. heterophilic) cooperation, which is to cooperate (defect) only with those who send the same message as they do. We show that under the replicator dynamics, homophilic cooperators engage in a battle of messages and will become dominated by whichever message that is most prevalent at the start, while populations of heterophilic cooperators exhibit a more harmonious behavior, converging to the state of the maximal diversity. Then we study the population shares of individuals associated with the four decision rules when prisoner's dilemma (PD) is taken as the base of the cheap-talk game. The hostility of heterophilics to individuals with similar tags leaves no room for pure cooperators to survive in a population of the two whereas the one-message dominance of the homophilics allows for pure cooperators with the same tag as the dominant homophilics to coexist in the population, demonstrating that in this context homophilics are more cooperative than heterophilics. By establishing a convergence theorem on strategies that are weakly dominated only in the absence of some other strategies, we complete previous research on PD games with preplay communication by proving that the frequencies of all but pure defectors converge to zero in a population where for each available pre-play message, there is a nonzero population share of defectors sending that message. This implies homophily and heterophily cannot maintain cooperation in this setting, which urges studying cheap-talk games under other reproduction dynamics.

5.1 Introduction

As we have discussed the effects of reciprocity on the evolution of cooperation in Chapter 4, here we focus on another mechanism known to promote cooperation, namely preplay communication. Evolutionary game theory has proven to be a powerful tool in understanding the emergence and

evolution of cooperation in natural and social systems [151, 106, 104, 23, 115, 140, 143, 149]. Perhaps the most widely studied game setup is the *prisoner's dilemma* (PD), in which defection is a strictly dominant strategy against cooperation despite the fact that mutual cooperation would result in higher payoffs than mutual defection for both players [112]. Researchers have struggled to explain how cooperation can actually be maintained and even promoted in evolutionary PD games. As a result, different modified game settings have been proposed, which include iterative [172, 135], spatial [116], multi-player [147] and emotional games [139]. In this context, a seemingly effective modification is the introduction of *cheap talk games* [46] that allows a costless, nonbinding, nonverifiable communication between the players before the game, a *preplay communication*. Players simultaneously send costless signals or messages to their opponents from a set available to each player before they play and consequently act based on the received messages [181]. Equivalently, in biology, one can think of individuals having recognizable phenotypes such as tags, on which they base their decisions [18, 174, 72]. Preplay communication differs from *signaling games* [122] where there is exactly one sender who does not act and only sends a message, and exactly one receiver who does not send a message and only acts according to the received message. A simple yet interesting setup in preplay communication is when each player treats similarly all received messages that are different from what she sends. This leads to four types of decision rules in a normal 2×2 game, two unconditional: *pure cooperation* (resp. *defection*), that is to always cooperate (defect), and two conditional: *homophilic cooperation* or *homophily* (resp. *heterophilic cooperation* or *heterophily*), that is to cooperate (defect) only with those sending the same message as you do.

Several studies have investigated the asymptotic behavior of individuals' population shares associated with these four decision rules under birth-death population dynamics. With regard to the evolution of homophily itself, some postulate homophily to evolve under a wide variety of conditions [48] while others claim that it evolves less easily than suggested by earlier studies [138]. Many others have studied the four types in a structured population and when PD is taken as the base game [80, 157, 59, 156, 60, 77]. Particularly in [59], homophilic cooperators are shown to take over 75 percent of a lattice-structured population. Jansson [74] argues that the dominance of homophilics in [59] is mainly due to the population structure not the tag mechanism (preplay communication). He upholds his claim by conducting a simulation on a well-mixed population with random interactions and 4 tags, and showing that homophilics only take 23 percent of the population whereas defectors take 61. However, still a considerable portion of the population is taken by non-pure defectors, leaving open the question of whether their survival, particularly that of homophilics, is due to random interactions and mutation. In general, it seems that preplay communication does not help to maintain cooperation in well-mixed populations. In [18] for example, it is stated that altruism is lost when the tag and decision rule traits are always inherited together, which is the case under usual evolutionary game theoretical settings such as the replicator dynamics [154, 135, 178]. The claim, however, lacks a mathematical proof and is explained only via examples and simulations. Mathematical analyses on the asymptotic behavior of population dynamics with preplay communication [19, 180, 79, 71] do not completely solve this problem either. We only know that if a strategy is strictly dominated in the base game of a cheap-talk game, e.g., cooperation in the PD game, its relative frequency goes to zero along any interior solution path of the replicator dynamics [181]. This implies that in the face of pure defectors, the population share of pure and homophilic cooperators goes to zero. So heterophilic cooperators are not ruled

out, pleading for a more comprehensive analysis.

In this chapter, we begin by uncovering the innate characteristic of well-mixed homophilic and heterophilic populations under the replicator dynamics in a general game setting. In particular, we find that populations of homophilic cooperators engage in a battle of messages (tags), and will become dominated by whichever message that was most prevalent at the start, while populations of heterophilic cooperators exhibit a more harmonious behavior, converging to a state of maximal message diversity. These results hold for the broad class of evolutionary games for which there is an advantage to mutual cooperation over defection. When engaged in a prisoner's dilemma game, a mixture of both types of cooperators can result in a limit cycle, with each tag dominating for some fraction of time. Moreover, pure cooperators may coexist with homophilic cooperators but vanish in the presence of heterophilics. Finally, by developing a convergence theorem on weakly dominated strategies, we establish once and for all that defectors completely take over the population, leaving no room for cooperation of any type including pure, homophilic and heterophilic cooperators.

5.2 Replicator dynamics for cheap-talk games

We consider a large population of individuals playing symmetric two-player games with two strategies, to "cooperate," denoted by C and to "defect," denoted by D, and with the payoffs integrated in the payoff matrix

$$\pi = \begin{array}{c} \\ \mathrm{C} \\ \mathrm{D} \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{C} & \mathrm{D} \end{array} \\ \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array}, \qquad T, R, P, S \in \mathbb{R},$$

where $R > P$, i.e., mutual cooperation exceeds mutual defection. We refer to this game as the *base game* G and are interested in when individuals have some preplay communication, resulting in a *cheap-talk game* $G_{\mathcal{M}}$, by which we mean the acts that before playing the base game G, each player sends one costless message to her opponent, and simultaneously receives the message sent by her opponent. The two preplay messages initiated by the two players are sent simultaneously and chosen from a finite set $\mathcal{M} = \{1, \ldots, m\}, m \geq 2$, of messages available to both players. The players may then base their strategies on the messages they have received, resulting in decision rules. We consider the case when messages different from that of the player herself are treated similarly, yielding the following four decision rules:

1. C, pure cooperation, under which the player always cooperates regardless of her opponent's message;

2. C^*, homophilic cooperation or homophily, under which the player cooperates if and only if her opponent's message matches that of her own;

3. D^*, heterophilic cooperation or heterophily, under which the player cooperates if and only if her opponent's message is different from that of her own;

4. D, pure defection, under which the player always defects regardless of her opponent's message.

Let $\mathcal{K} = \{C, C^*, D^*, D\}$ be the set of decision rules. Then each individual sends a message $i \in \mathcal{M}$ and follows a decision rule $X \in \mathcal{K}$, resulting in a cheap-talk pure strategy X_i that we characterize by the unit vector whose elements are all zero except for the $(4(i-1)+p_X)^{\text{th}}$ element:

$$X_i = [\underbrace{0 \quad \dots \quad 0 \quad 1}_{4(i-1)+p_X} 0 \quad \dots 0]^T$$

where $p_X, X \in \mathcal{K}$, is defined as $p_C = 1,\ p_{C^*} = 2,\ p_{D^*} = 3$ and $p_D = 4$. Based on their strategies in the cheap-talk game, players earn payoffs in the base game, which is captured by the *cheap-talk payoff matrix* $\pi_{\mathcal{M}}$ defined by

$$\pi_{\mathcal{M}} = \begin{pmatrix} \pi \otimes \mathbf{1} & \mathbf{1} \otimes \pi & \dots & \mathbf{1} \otimes \pi \\ \mathbf{1} \otimes \pi & \pi \otimes \mathbf{1} & \dots & \mathbf{1} \otimes \pi \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} \otimes \pi & \mathbf{1} \otimes \pi & \dots & \pi \otimes \mathbf{1} \end{pmatrix}_{4m \times 4m},$$

where $\otimes$ denotes the Kronecker product and $\mathbf{1}$ the 2×2 all-one matrix. Then the payoff an individual playing strategy X_i, $X \in \mathcal{K}, i \in \mathcal{M}$, earns against another individual playing strategy Y_j, $Y \in \mathcal{K}, j \in \mathcal{M}$, equals $u(X_i, Y_j)$ where u, the *utility function*, is defined by $u(x, y) = x^T \pi_{\mathcal{M}} y$ for $x, y \in \mathbb{R}^{4m}$.

We study the evolution of individuals' population shares under the replicator dynamics. Denote the population share of individuals playing strategy X_i by x_{X_i}. Then the population state vector equals

$$x = \sum_{X \in \mathcal{K}, i \in \mathcal{M}} x_{X_i} X_i,$$

and the average payoff an x_{X_i}-playing individual earns against the population equals $u(X_i, x)$. The evolution over time of the population share of individuals playing X_i is given by the replicator dynamics

$$\dot{x}_{X_i} = [u(X_i, x) - u(x, x)]\, x_{X_i}, \quad i = 1, \dots, n. \tag{5.1}$$

The main goal of this letter is to study the asymptotic behavior of the four types of individuals in different population mixtures under the dynamics (5.1).

5.3 Convergence results

We provide the convergence results in this section. First, we consider a population of homophilic cooperators and then that of heterophilic cooperators.

Theorem 5.1. *Consider an exclusive population of homophilic cooperators under the dynamics* (5.1)*. Then for any* $i, j \in \mathcal{M}$*,*

$$x_{C_i^*}(0) > x_{C_j^*}(0) \Rightarrow \lim_{t \to \infty} x_{C_j^*}(t) = 0.$$

Proof. Should $x_{C_j^*} = 0$, the result would be trivial. So assume otherwise. Then $0 < x^0_{C_j^*} < 1$, and in fact following some standard continuity argument, one can check that $0 < x_{C_j^*}(t) < 1$ for all $t \geq 0$. So we can define the ratio $\frac{x_{C_i^*}}{x_{C_j^*}}$ of the population shares and calculate its time derivative

$$\frac{d}{dt}\left(\frac{x_{C_i^*}}{x_{C_j^*}}\right) = [u(x_{C_i^*}, x) - u(x, x)]\frac{x_{C_i^*}}{x_{C_j^*}}$$
$$\Rightarrow \frac{d}{dt}\left(\frac{x_{C_i^*}}{x_{C_j^*}}\right) = (R - P)[x_{C_i^*} - x_{C_j^*}]\frac{x_{C_i^*}}{x_{C_j^*}}. \tag{5.2}$$

Since $R > P$ and $x_{C_i^*}(0) > x_{C_j^*}(0)$, the right-hand side of (5.2) is positive at $t = 0$, so $x_{C_i^*}/x_{C_j^*}$ increases at t_0, which in turn makes $x_{C_i^*} - x_{C_j^*}$ increase and so does the infinitesimal increment of $x_{C_i^*}/x_{C_j^*}$. Repeating this argument, we know that $x_{C_i^*}/x_{C_j^*}$ keeps increasing nontrivially for the entire evolution since $x_{C_i^*}$ and $x_{C_j^*}$ are well defined and $x_{C_j^*} > 0$ for all t. So

$$\lim_{t\to\infty} \frac{x_{C_i^*}(t)}{x_{C_j^*}(t)} = \infty.$$

On the other hand, since $0 < x_{C_i^*}, x_{C_j^*} < 1$, this implies $\lim_{t\to\infty} x_{C_j^*}(t) = 0$, which completes the proof. □

As indicated by Theorem 5.1, homophilic's hostility towards outsiders results in a *battle of messages (tags)*, where a single message ends up taking over the entire population (See Fig. 5.1). Not surprisingly, the winning message turns out to be the one that started with the greatest

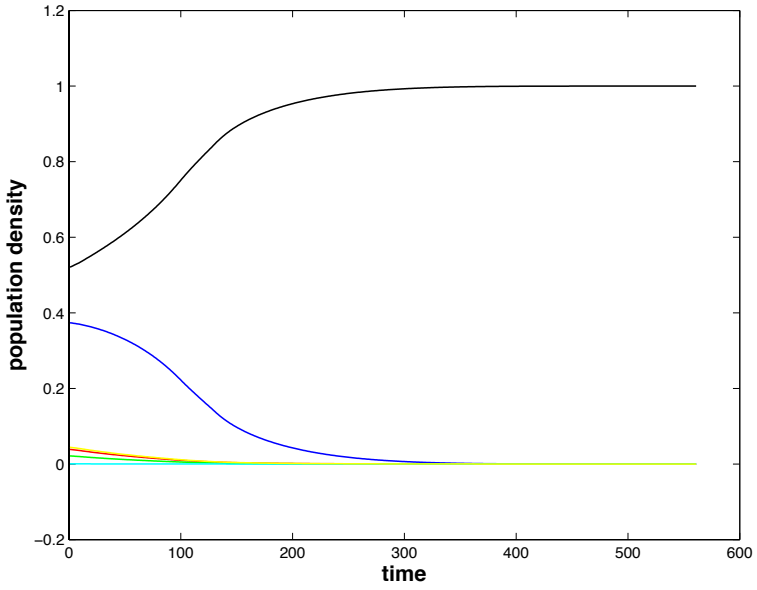


Figure 5.1: Evolution of homophilic cooperators.

population share. The intuition behind is that individuals sending the most populous message are also most likely to interact with cooperative opponents. The final population state is also robust against message mutations, meaning that any newly formed messages are quickly eliminated. In the rare case of having two or more most populous message in the start, they will share the final population equally.

Theorem 5.2. *Consider an exclusive population of heterophilic cooperators under the dynamics* (5.1). *Then for any* $i, j \in \mathcal{M}$,

$$x_{D_i^*}(0) \neq 0, x_{D_j^*}(0) \neq 0 \Rightarrow \lim_{t\to\infty} x_{D_i^*}(t) = \lim_{t\to\infty} x_{D_j^*}(t).$$

Proof. Similar to (5.2) we obtain

$$\frac{d}{dt}\left(\frac{x_{D_i^*}}{x_{D_j^*}}\right) = -(R-P)[x_{D_i^*} - x_{D_j^*}]\frac{x_{D_i^*}}{x_{D_j^*}}.$$

One can check that the solution $x_{D_i^*} = x_{D_j^*}$ is globally asymptotically stable in the zone $0 < x_{D_i^*} = x_{D_j^*} < 1$ under the above differential equation, which completes the proof. □

Remark 5.1. *Indeed there is a compelling one-to-one correspondence between the dual trajectories of the replicator dynamics starting from* $\Delta_{\mathcal{M}}(C^*)$ *and* $\Delta_{\mathcal{M}}(D^*)$ *respectively: they can be taken as the same trajectory, one flows forwards and the other backwards in time.*

From Theorem 5.2, in populations consisting exclusively of heterophilic cooperators, we observe the opposite phenomenon as that in the homophilic case, resulting in a *harmony of messages (tags)* (See Fig. 5.2). Although diversity is a known property of heterophilic populations, it has not yet

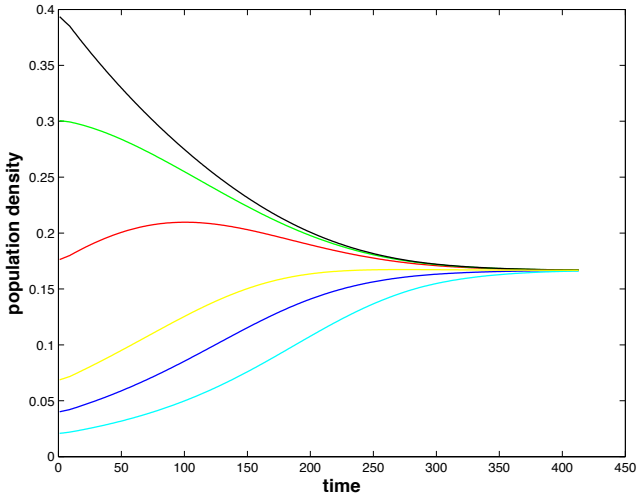


Figure 5.2: Evolution of heterophilic cooperators.

been shown that heterophily indeed drives the population to a *maximally balanced* state with all messages holding equal shares of the population. This occurs because members sending the

least common message are now most likely to meet a cooperative opponent while the converse is true for members sending the most common message. Hence there is a balancing effect putting upward pressure on those tags having below average representation and downward pressure on those having above average representation until the difference in population share between groups converges to zero. An interesting side effect occurs if a brand new message appears via mutation. This new message will be welcomed into the population but will grow only until it reaches the same population share as all the other tags. In this sense heterophilic cooperators produce a population that is more tolerant than homophilic cooperators and will hence result in greater diversity. These results hold for any symmetric game in which the payoffs for mutual cooperation are greater than those for mutual defection. This battle and harmony of messages has been reported in the simulation work [138], but in a death-birth process and different game setup.

To investigate how cooperative homophilics and heterophilics are, particularly in the face of pure cooperators and defectors, we choose the base game to be the challenging prisoner's dilemma game, in which the payoffs satisfy

$$T > R > P > S. \tag{5.3}$$

So far we have seen that a population of homophilics will become dominated by a single tag, while a population of heterophilics converges to a maximally tag-balanced state. When these two types are mixed, one would expect a conflict to arise out of the simultaneous battle and harmony of messages, and indeed we observe just that - the population share of each tag exhibits large oscillations, which, depending on the payoffs and the number of available tags, may persist indefinitely suggesting a heteroclinic cycle (See Fig. 5.3), or instability may occur. For example,

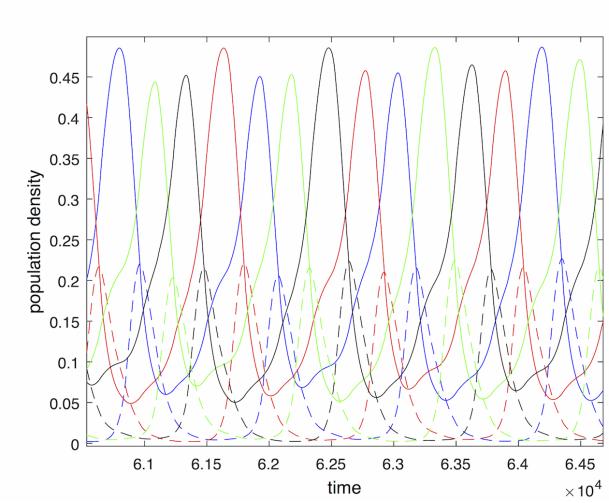


Figure 5.3: Co-evolution of homophilic and heterophilic cooperators.

consider a homophilic cooperator with some rare message i appearing as a mutant in a population. This mutant will defect against all messages not equal to i, which hold for the vast majority of the population. Then, if the average individual payoff in the population is below some threshold, the mutant will invade and eventually wipe out all existing messages, resulting in a uniform population of homophilics sending message i. Now a heterophilic cooperator sending the same message i can invade this population since it defects against its own tag, but the homophilics that make up the

population cooperate in return with this new mutant. As a result, the population will eventually be completely taken over by heterophilics with message i, which again creates an opening for an homophilic cooperator with a different rare tag j to invade, and the process repeats indefinitely among different messages, resulting in a heteroclinic cycle.

Now if we consider a population that contains both homophilic and pure cooperators, an interesting phenomenon occurs. Having various messages but no preference in others, the pure cooperators are subject to the prejudices of the other homophilic cooperators and thus the battle of messages remains in effect. Once again, the message(s) occupying the largest portion of the initial population will prevail as indicated in the following theorem, and the final population will generally contain both homophilic and pure cooperators sending this message. Let $x_X, X \in \mathcal{K}$, and $x_i, i \in \mathcal{M}$, denote the population share of individuals following decision rule X and sending message i, respectively. Clearly $x_X = \sum_{j\in\mathcal{M}} x_{X_j},\ x_i = \sum_{Y\in\mathcal{K}} x_{Y_i}$.

Theorem 5.3. *Consider an exclusive population of pure and homophilic cooperators where for every message $i \in \mathcal{M}$, $x_{C_i^*} > 0$. Then under the dynamics (1) and when (2) is fulfilled, at least one of the followings holds*

$$\lim_{t\to\infty} x_C(t) = 0$$

or

$$\exists i \in \mathcal{M} : \lim_{t\to\infty} x_i(t) = 1.$$

Proof. A typical cheap-talk payoff matrix would be

$$\begin{array}{c} \\ C_1 \\ C_1^* \\ D_1^* \\ D_1 \\ \\ C_2 \\ C_2^* \\ D_2^* \\ D_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} C_1 & C_1^* & D_1^* & D_1 \end{array} \\ \left(\begin{array}{cccc|cccc} R & R & S & S & R & S & R & S \\ R & R & S & S & T & P & T & P \\ T & T & P & P & R & S & R & S \\ T & T & P & P & T & P & T & P \\ - & - & - & - & - & - & - & - \\ R & S & R & S & R & R & S & S \\ T & P & T & P & R & R & S & S \\ R & S & R & S & T & T & P & P \\ T & P & T & P & T & T & P & P \end{array}\right) \\ \begin{array}{cccc} C_2 & C_2^* & D_2^* & D_2 \end{array} \end{array}. \tag{5.4}$$

Based on the population in question, $x(0) \in \mathrm{int}(\Delta(\mathcal{S}))$ for some $\mathcal{S} \subseteq \{X_j \,|\, X \in \{C, C^*\}, j \in \mathcal{M}\}$ where $\mathcal{S} \ni C_i^*$ for all $i \in \mathcal{M}$. Consider the reduced cheap-talk payoff matrix corresponding to decision rules C and C^*. A typical case would be

$$\begin{array}{c} \\ C_1 \\ C_1^* \\ \\ C_2 \\ C_2^* \end{array}\!\!\begin{array}{c} \begin{array}{cc} C_1 & C_1^* \end{array} \\ \left(\begin{array}{cc|cc} R & R & R & S \\ R & R & T & P \\ - & - & - & - \\ R & S & R & R \\ T & P & R & R \end{array}\right) \\ \begin{array}{cc} C_2 & C_2^* \end{array} \end{array}.$$

For any $i \in \mathcal{M}$, C_i is weakly dominated by C_i^* in $\Delta(\mathcal{S})$. In addition, for any $j \in \mathcal{M}, j \neq i$,

$$u(C_i^*, C_j^*) > u(C_i, C_j^*) \qquad \text{and} \qquad u(C_i^*, C_j) > u(C_i, C_j).$$

Hence, in view of Proposition 5.2 in the Appendix section, at least one of the followings holds:

$$\lim_{t\to\infty} x_{C_i}(t) = 0$$

or

$$\lim_{t\to\infty} x_{C_j}(t) = \lim_{t\to\infty} x_{C_j^*}(t) = 0.$$

Repeating this argument for all other $j \in \mathcal{M}, j \neq i$, we obtain

$$\lim_{t\to\infty} x_{C_i}(t) = 0 \tag{5.5}$$

or

$$\lim_{t\to\infty} x_{C_j}(t) = \lim_{t\to\infty} x_{C_j^*}(t) = 0 \quad \forall j \in \mathcal{M} - \{i\} \;\Rightarrow\; \lim_{t\to\infty} x_i(t) = 1. \tag{5.6}$$

Since at least one of (5.5) and (5.6) takes place for every $i \in \mathcal{M}$, we conclude that at least one of the following two will happen

$$\forall i \in \mathcal{M} \quad \lim_{t\to\infty} x_{C_i}(t) = 0$$

or

$$\exists i \in \mathcal{M} : \lim_{t\to\infty} x_i(t) = 1,$$

which completes the proof. □

In fact, homophilic cooperators are the only type with which pure cooperators can coexist in the prisoner's dilemma. Heterophilic cooperators still defect against pure cooperators sending the same message, so in a mixed population of these two types engaged in a prisoner's dilemma, the heterophilic cooperators will wipe out the pure cooperators as they converge to the balanced state:

Theorem 5.4. *Consider an exclusive population of pure and heterophilic cooperators where for every message $i \in \mathcal{M}$, if $x_{C_i} > 0$, then $x_{D_i^*} > 0$. Then under the dynamics (1) and when (2) is fulfilled,*

$$\lim_{t\to\infty} x_C(t) = 0.$$

Proof. Based on the population in question, $x(0) \in \mathrm{int}(\Delta(\mathcal{S}))$ for some $\mathcal{S} \subseteq \{X_j \,|\, X \in \{C, D^*\}, j \in \mathcal{M}\}$ where $\mathcal{S} \ni D_i^*$ if $\mathcal{S} \ni C_i$, for all $i \in \mathcal{M}$. Consider the reduced cheap-talk payoff matrix corresponding to decision rules C and D^*. A typical case is

$$\begin{array}{c} \\ C_1 \\ D_1^* \\ \\ C_2 \\ D_2^* \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & D_1^* & & C_2 & D_2^* \end{array} \\ \left(\begin{array}{cccccc} R & S & | & R & R \\ T & P & | & R & R \\ - & - & - & - & - \\ R & R & | & R & S \\ R & R & | & T & P \end{array}\right) \end{array}.$$

For any $i \in \mathcal{M}$, C_i is weakly dominated by D_i^* in $\Delta(\mathcal{S})$. In addition, for any $j \in \mathcal{M}, j \neq i$,

$$u(D_i^*, C_j) > u(C_i, C_j).$$

Hence, in view of Proposition 5.2,

$$\lim_{t\to\infty} x_{C_j}(t) = 0$$

The proof then follows the fact that the above equation holds for every $i \in \mathcal{M}$. □

It is worth mentioning that pure cooperators also do not survive in a population of the three types of individuals, as stated in the following.

Proposition 5.1. *Consider an exclusive population of pure, homophilic and heterophilic cooperators where for every message $i \in \mathcal{M}$, if $x_{C_i} > 0$, then $x_{D_i^*} > 0$. Then under the dynamics (1) and when (2) is fulfilled,*

$$\lim_{t\to\infty} x_C(t) = 0.$$

Proof. Based on the population in question, $x(0) \in \text{int}(\Delta(\mathcal{S}))$ for some $\mathcal{S} \subseteq \{X_j \,|\, X \in \{C, D^*\}, j \in \mathcal{M}\}$ where $\mathcal{S} \ni D_i^*$ if $\mathcal{S} \ni C_i$, for all $i \in \mathcal{M}$. Consider the reduced cheap-talk payoff matrix corresponding to decision rules C and D^*. A typical case would be

$$\begin{array}{c} \\ C_1 \\ C_1^* \\ D_1^* \\ \\ C_2 \\ C_2^* \\ D_2^* \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_1^* & D_1^* \end{array} \\ \left(\begin{array}{ccc|c|ccc} R & R & S & & R & S & R \\ R & R & S & & T & P & T \\ T & T & P & & R & S & R \\ - & - & - & - & - & - & - \\ R & S & R & & R & R & S \\ T & P & T & & R & R & S \\ R & S & R & & T & T & P \end{array}\right) \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\

Theorem 5.6. *Let $\mathcal{P}$ be a set of pure strategies, and consider a nonempty subset $\mathcal{H}$ of it. Suppose that a pure strategy a is weakly dominated by some strategy y in the face $\Delta(\mathcal{H})$. Also let $u(y,b) > u(a,b)$ for a pure strategy $b \in \mathcal{H}$. If*

$$\lim_{t\to\infty} x_j = 0 \qquad\qquad \forall j \in \mathcal{P} - \mathcal{H}, \tag{5.7}$$

then under the dynamics (1) and for any $x(0) \in \operatorname{int}(\Delta(\mathcal{P}))$, it holds that

$$\lim_{t\to\infty} x_a = 0 \quad \vee \quad \lim_{t\to\infty} x_b = 0. \tag{5.8}$$

Proof. Some ideas from the proof of Proposition 3.2 in [181] are used here. For each pure strategy j, let $\gamma_j = u(y - e^a, e^j)$. Since a is weakly dominated by y in $\Delta(\mathcal{H})$, it holds that $\gamma_j \geq 0, \forall j \in \mathcal{H}$. Moreover, $\gamma_b > 0$ because of $u(y,b) > u(a,b)$. Define the function $v_a : \operatorname{int}(\Delta(\mathcal{P})) \to \mathbb{R}$ by

$$v_a(x) := \log(x_a) - \sum_{j\in\mathcal{P}} y_j \log(x_j).$$

Clearly v_a is differentiable and its time derivative equals

$$\begin{aligned}
\frac{d}{dt}v_a(x) &= \frac{\dot{x}_a}{x_a} - \sum_{j\in\mathcal{P}} \frac{y_j \dot{x}_j}{x_j} \\
&= u(e^a - x, x) - \sum_{j\in\mathcal{P}} y_j u(e^j - x, x) \\
&= u(e^a - x, x) - u(y - x, x) = -u(y - e^a, x) \\
&= -\sum_{j\in\mathcal{P}} u(y - e^a, e^j) x_j = -\sum_{j\in\mathcal{P}} \gamma_j x_j \\
&= -\sum_{j\in\mathcal{H}\cap\mathcal{P}} \gamma_j x_j - \sum_{j\in\mathcal{P}-\mathcal{H}} \gamma_j x_j \\
&\leq -\gamma_b x_b - \sum_{j\in\mathcal{P}-\mathcal{H}} \gamma_j x_j.
\end{aligned}$$

Hence,

$$v_a(x) \leq v_a(x^0) - \gamma_b \operatorname{int}_0^t x_b(\tau)d\tau + \sum_{j\in\mathcal{P}-\mathcal{H}} \gamma_j \operatorname{int}_0^t x_j(\tau)d\tau.$$

Because of (5.7) and the fact that all $x_j(t)$s are confined to $[0,1]$ for all $t \geq 0$, each integral $\operatorname{int}_0^t x_j(\tau)d\tau, j \in \mathcal{P} - \mathcal{H}$, converges to some constant α_j. On the other hand, since $0 \leq x_b(t) \leq 1$ for all $t \geq 0$, the integral $\operatorname{int}_0^t x_b(\tau)d\tau$ either converges to $+\infty$ or to some constant α_b. In the first case, since $\gamma_b > 0$ it holds that $v_a \to -\infty$ and hence $x_a \to 0$, according to the definition of v_a. In the second case, by the uniform continuity of x_b it can be shown that $x_b \to 0$, which completes the proof. □

Now we proceed to the proof of Theorem 5.5.

Proof of Theorem 5.5. Based on the population in question, $x(0) \in \text{int}(\Delta(\mathcal{S}))$ for some $\mathcal{S} \subseteq \{X_j \,|\, X \in \mathcal{K}, j \in \mathcal{M}\}$ where $\mathcal{S} \ni D_i$ for all $i \in \mathcal{M}$. Consider the reduced cheap-talk payoff matrix corresponding to decision rules C and D^*. A typical case is presented at (5.4). For any $i \in \mathcal{M}$, C_i is weakly dominated by D_i in $\Delta(\mathcal{S})$. In addition,

$$u(D_i, C_i) > u(C_i, C_i).$$

Hence, in view of Proposition 5.2,

$$\lim_{t\to\infty} x_{C_i}(t) = 0 \quad \forall i \in \mathcal{M} \tag{5.9}$$

Similarly, for any $i \in \mathcal{M}$, C_i^* is weakly dominated by D_i in $\Delta(\mathcal{S})$. In addition,

$$u(D_i, C_i^*) > u(C_i^*, C_i^*).$$

Hence, in view of Proposition 5.2,

$$\lim_{t\to\infty} x_{C_i^*}(t) = 0 \quad \forall i \in \mathcal{M} \tag{5.10}$$

Now define the set $\bar{\mathcal{S}} = \mathcal{S} - \{X_i \,|\, X \in \{C, C^*\}, i \in \mathcal{M}\}$. For any $i \in \mathcal{M}$, there exists a sufficiently small $\epsilon > 0$ such that the mixed strategy $y = (1-\epsilon)D_i + \epsilon D_j$ for some $j \neq i$ weakly dominates D_i^* in $\Delta(\bar{\mathcal{S}})$. In addition,

$$u(y, D_i^*) > u(D_i^*, D_i^*) \qquad \forall i \in \mathcal{M}.$$

Hence, by taking $\mathcal{P} = \mathcal{S}$ and $\mathcal{H} = \bar{\mathcal{S}}$, from Theorem 6 we obtain

$$\lim_{t\to\infty} x_{C_i^*}(t) = 0 \quad \forall i \in \mathcal{M}. \tag{5.11}$$

By summarizing (5.9) to (5.11), we arrive at the proof. □

5.4 Concluding remarks

The results in this chapter reveal the intrinsic qualities of homophilic and heterophilic cooperators. "Only my phenotype (message) should survive" is the tendency homophilics show leading indeed to only one group surviving in their exclusive population. This suggests the presence of homophily in populations where one phenotype dominates the population share. It is also because of this attitude that homophilics are the only type of individuals with whom pure cooperators may survive, under the prisoner's dilemma. On the other hand, heterophilics exhibit a"welcoming" attitude leading to diversity and balance in their exclusive population. This suggests the presence of heterophily in highly phenotype-balanced populations. A population mixture of these two types results in oscillations that can either persist as in a heteroclinic cycle, or become unstable. This heteroclinic cycle may be helpful when seeking populations exhibiting both homophily and heterophily. The last result here is that if pure defectors are present, there is no room for cooperation of any kind in the prisoners' dilemma.

5.5 Appendix

We present a classical result on weakly dominated strategies in this section [181]. Consider a normal symmetric two-player game with n strategies and a payoff matrix $A \in \mathbb{R}^{n\times n}$. Define the simplex

$$\Delta := \left\{ x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = 1, 0 \leq x_i \leq 1 \right\}.$$

Let $e^i, i = 1 \ldots, n$, be the nth column of the $n \times n$ identity matrix, and define $\mathcal{E} = \{e^i | i = 1, \ldots, n\}$. Each e^i is a pure strategy. A *mixed strategy* is a vector $x \in \Delta$, and can be obtained by some convex combination of the vertices of Δ. Consider a nonempty subset $\mathcal{H} \subseteq \mathcal{E}$. The convex hull of $\mathcal{H}$ is a *face* of Δ and is denoted by $\Delta(\mathcal{H})$ [181]. We say that strategy x *weakly dominates strategy* y *in* $\Delta(\mathcal{H})$, if (i) $x, y \in \Delta(\mathcal{H})$, (ii) $u(x, z) \geq u(y, z)$ for all $z \in \Delta(\mathcal{H})$ and (iii) $u(x, r) > u(y, r)$ for at least one $r \in \Delta(\mathcal{H})$. This is a reformulation of weak dominance in the whole simplex. Besides being a strategy, a vector $x \in \Delta$ also can also represent a population vector whose entry x_i denote the population share of individuals playing strategy i. Then the replicator dynamics can be written as

$$\dot{x}_i = \left[u(e^i, x) - u(x, x)\right] x_i \qquad i = 1, \ldots, n. \tag{5.12}$$

For ease of analysis, we propose the following notation. Given a pure strategy $v = e^i$ for some $i \in \{1, \ldots, n\}$ define $x_v := x_i$. For example, when $v = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, we have that $x_v = x_2$. The following result, which we have reformulated to make it applicable to $\Delta(\mathcal{H})$ instead of just Δ, is a classical result on convergence of weakly dominated strategies under the dynamics (5.12).

Proposition 5.2 ([181] weakly dominated strategies)**.** *Suppose that a pure strategy a is weakly dominated by some strategy y in $\Delta(\mathcal{H})$. If $u(y, b) > u(a, b)$ for some pure strategy b, then the following holds under the dynamics* (5.12) *and for any initial state $x(0) \in \operatorname{int}(\Delta(\mathcal{H}))$:*

$$\lim_{t\to\infty} x_a(t) = 0 \quad \vee \quad \lim_{t\to\infty} x_b(t) = 0.$$

Part II

Finite Well-Mixed Populations

Chapter 6

Asynchronous decision-making dynamics under best-response update rule in finite heterogeneous populations

To study how sustainable cooperation might emerge among self-interested interacting individuals, we investigate the long-run behavior of the decision-making dynamics in a finite, well-mixed population of individuals, who play collectively over time a population game. Repeatedly each individual is activated asynchronously to update her decision either to cooperate or defect according to the myopic best-response rule. The game's payoff matrices, chosen to be those of either prisoner's dilemma or snowdrift games to underscore cooperation-centered social dilemmas, are fixed, but can be distinct for different individuals. So the overall population is heterogeneous. We first classify such heterogeneous individuals into different types according to their cooperating tendencies stipulated by their payoff matrices. Then we show that no matter what initial strategies the individuals decide to use, surprisingly one can always identify one type of individuals as a benchmark such that after a sufficiently long but finite time, individuals more cooperative compared to the benchmark almost surely always cooperate while those less cooperative compared to the benchmark almost surely defect. When such fixation takes place, the total number of cooperators in the population either becomes fixed or fluctuates at most by one. Such insight provides theoretical explanation for some complex behavior recently reported in simulation studies that highlight the puzzling effect of individuals' heterogeneity on collective decision-making dynamics. Then we demonstrate how to control the total number of cooperators in the long run by influencing the agents' payoffs gained from the games over time. In particular, given a population and limited number of available payoff matrices that the agents may be associated with, we determine all reachable final number of cooperators, and also all payoff matrix distributions among the agents that lead to those final numbers. In addition, we find the minimum number of agents whose payoff matrices must be changed to reach those desired numbers of cooperators in the long run. The results indicate that the minimum number does not equal the difference of the current number of cooperators in the population from the desired one, a side-effect caused by the level-off phenomenon that shows up in such models.

6.1 Introduction

We have studied how cooperation evolves in infinite populations governed by the replicator dynamics in Part I; in this part, we investigate the evolution of cooperation in finite well-mixed populations. We focus on the case when the individuals play the snowdrift or prisoner's dilemma games in this Chapter and focus on the coordination game in Chapter 7. The study on mechanisms of emergence of cooperation in self-interested social populations has attracted extensive attention in the past decades [62, 66, 68, 13, 118]. One general consensus is that human's ability to learn plays a key role to build up sustainable collective cooperation in a competitive environment. Recent experimental studies [176, 107, 169] indicate that in a social group how human subjects learn to update their strategies may affect dramatically the outcome of the group's interactions. For example, the cooperation level of a group of individuals is higher when each individual focuses more on the frequencies of the behaviors of the peers (frequency-based learning) instead of the success among the peers (success-based learning) [176]. However, how the frequency-based individuals learn remains an open question, which requires much more in-depth theoretical investigation on possible learning rules and the resulted population decision-making dynamics. Evolutionary game theory [51, 10, 58, 189, 144, 91] has provided promising theoretical tools, and in fact postulated a variety of dynamical models under frequency-based learning [170, 166, 138, 49, 135]. In one popular model, decision makers always follow the myopic best response rule to maximize their utilities against their opponents [149, 155, 39].

Various simulation based studies have disclosed several features of such best-response evolutionary dynamics. For homogeneous populations where individuals have the same tendency to cooperate, people have identified the surprising suppression of network effects in different game setups [164, 61, 146]. For heterogeneous populations where individuals' tendencies to cooperate are different, we have reported a new "level-off" phenomenon [139]; to be more specific, *level-off* here refers to the phenomenon that staring with a population of a low cooperation level, the level rises with the share of more-cooperative individuals but then levels off as the share reaches some threshold. These and other related simulation studies [99] provide intuition on how the decision-making dynamics in large populations governed by the best-response rule evolve over time, but sometimes the results are descriptive due to the lack of rigorous mathematical proofs. Some related mathematical results can in general only be applied to homogeneous populations in specific setups, such as deterministic updates within well-mixed and finite populations [7], infinite population or mean field approximation [110, 88, 92, 123], and noisy updates [187, 84, 9, 109]. Only a few papers with mathematical analysis have studied heterogeneous populations [2, 70], but still with restrictive assumptions like infinite populations or deterministic updates. So a great challenge is to develop new mathematically rigorous tools to analyze asynchronous decision-making dynamics under the best-response update rule in finite heterogeneous populations.

While to model and analyze such population dynamics are the necessary steps towards understanding the evolution of cooperation, controlling them is a more relevant topic for its interesting applications, e.g., facilitating the spread of innovation and technology [187]. As with most population dynamics in general, control of the strategic interactions under the best-response update rule is quite challenging. For example, it is already a non-trivial task to identify the most appropriate control input. There are a number of options, e.g., individuals' utilities, the strategies of a group of individuals and the addition of some individuals with fixed strategies. The problem becomes even

more complicated in the case of heterogeneous populations since the heterogeneity distribution among the individuals may prevent a uniform choice of a control input. If individuals' strategies are chosen as the control input, it remains a demanding task to decide which group of individuals must be targeted in order to e.g., minimize the control efforts. Different methods have been suggested to control the number of agents with a specific strategy in decision-making populations. For example, in [145, 144], the strategies of a group of agents are fixed to a desired strategy for the duration of the game; in [91], the payoffs of a stochastic snowdrift game are changed; and in [186], some changes in terms of an emission tax rate and the price of an emission permit are applied to individuals' utilities. However, control of best-response dynamics, specially in heterogeneous populations, remains an open problem.

In this chapter we consider a finite, well-mixed, heterogeneous population in which each individual, also referred to as agent, has her own (possibly unique) utility function when playing evolutionary games where the base game is the classical prisoner's dilemma or snowdrift game to capture the well-known cooperation social dilemma. The agents are activated randomly to play the game and consequently best-response updates are asynchronous. Our main results are to explicitly determine the long run dynamic behavior without using mean-field or other approximation methods and to control the dynamics. We are able to show that no matter what initial strategies the individuals decide to use, surprisingly one can always identify one type of individuals as a benchmark such that after a sufficiently long but finite time, more-cooperative individuals almost surely cooperate while those less cooperative almost surely defect. It is also shown that the total number of cooperators in the population will either become fixed or fluctuate at most by one. The importance of these new findings is threefold. First, these global convergence results explain rigorously why sustainable cooperation can emerge among selfish heterogeneous individuals. Second, the level-off phenomenon [139] is validated and proved rigorously and thus shows the subtlety of controlling the level of cooperation if one wants to use population heterogenity as a means to intervene collective decision-making processes. And third, the different thresholds in the level-off behavior for populations with diverse compositions indicate the sometimes restrictive nature of the homogeneity assumption in social evolutionary models.

After presenting the convergence results, we provide a control mechanism for the population dynamics. While it may be possible to set a robot to a desired strategy, it seems unrealistic to ask a company to always cooperate with other competing firms or to ask a person to always expose the same behavior in face of other members of a social group. Instead we follow the approach where payoff incentives are provided to a group of agents, resulting in changes to their payoff matrices. This change can represent increasing the taxes in case of companies, effective training in case of human or a simple modification in the programmed values in case of robots. Then for a given population, we first find those numbers of cooperators in the long run that one can reach by modifying the agents' payoffs. Next, for a desired final number of cooperators, we find m, the minimum number of agents whose payoff matrices must be changed to reach that number, and show which agents' payoff matrices must be changed accordingly. The results indicate that the minimum number m is not necessarily the difference of the current number of cooperators in the population from the desired one, a side-effect caused by the level-off phenomenon.

The rest of the chapter is organized as follows. In Section 6.2, we present the model for decision-making dynamics. The main convergence results are shown in Section 6.3. In Section 6.4, we focus on the mathematical characteristics of the level-off phenomenon. Then in Section 6.5, we present

the control mechanism for the population dynamics. We finalize the chapter with the conclusions in Section 6.6.

6.2 Decision-making under best-response updates

We consider a finite, well-mixed population of n agents that are participating in a population game evolving over time $t = 0, 1, \ldots$. Each agent can choose either to cooperate (C) or defect (D). At each time t, an agent is randomly activated to update her strategy according to how well she is doing when she plays her current strategy against the population. More specifically, the four possible payoffs of an agent i, $i = 1, \ldots, n$, are summarized in the 2×2 payoff matrix

$$A_i = \begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} R_i & S_i \\ T_i & P_i \end{pmatrix} \end{array}, \tag{6.1}$$

where the payoffs R_i, T_i, S_i and P_i are real numbers corresponding to strategy pairs C-against-C, D-against-C, C-against-D and D-against-D respectively. Let $s_i(t)$ denote agent i's strategy at time t in the vector form, which is either $s^C \triangleq [1\ 0]^\top$ to cooperate or $s^D \triangleq [0\ 1]^\top$ to defect. Obviously, $s^C = \mathbf{1} - s^D$ with $\mathbf{1} = [1\ 1]^\top$. So agent i's payoff at time t against the population can be calculated by

$$u_i\left(s_i(t), s_C(t)\right) = s_i(t)^\top A_i\, s_C(t)$$

where $s_C(t) \triangleq \begin{bmatrix} n_C(t) & n - n_C(t) \end{bmatrix}^\top$ and $n_C(t)$ denotes the number of cooperators in the whole population at time t. The myopic best-response update rule for agent i dictates that agent i chooses the strategy that maximizes her payoff. In case both cooperation and defection return the same payoff, we assume agent i sticks to her current strategy. Therefore, the update rule is that agent i sticks to her current strategy only if her alternative strategy does not give her a higher payoff, namely

$$s_i(t+1) = \begin{cases} s_i(t) & \text{if } u_i(s_i(t), s_C(t)) \geq u_i(\mathbf{1} - s_i(t), s_C(t)) \\ \mathbf{1} - s_i(t) & \text{otherwise} \end{cases}. \tag{6.2}$$

When studying homogeneous populations, people are mainly interested in how the portion of cooperators changes over time; for heterogeneous populations like in this chapter, the more complex central topic is how the portions of cooperators with different utility functions evolve. Towards this end, we classify the heterogeneous individuals into different types according to their cooperating tendencies stipulated by their payoff matrices A_i in (6.1). We focus on those A_i with special structures [139]: the entries of A_i satisfy

$$T_i > R_i > \max\{S_i, P_i\}, \qquad S_i \neq P_i. \tag{6.3}$$

Then A_i corresponds to either a prisoner's dilemma (PD) game satisfying $P_i > S_i$ or a snowdrift (SD) game satisfying $S_i > P_i$. We call each agent with a PD (resp. SD) payoff matrix a *PD agent* (resp. *SD agent*). In fact, as we will later show in Lemma 6.1, PD agents always tend to

defect under (6.2); however, an SD agent's tendency to cooperate depends on the ratio $\frac{S_i-P_i}{T_i-R_i+S_i-P_i}$, namely, the higher the ratio is, the more cooperative the agent becomes, except for those ratios in the same interval $(\frac{k}{n}, \frac{k+1}{n})$ for some $k \in \{0, 1, \ldots, n-1\}$ that result in the same cooperative tendency. For this reason, we categorize all those SD players whose ratios are within the same interval $(\frac{k}{n}, \frac{k+1}{n})$ into the same *type*. We assume there are altogether $l > 0$ types and label them by $1, \ldots, l$ according to the *descending* order of the ratios. Correspondingly, each SD agent of the jth, $j \in \{1, \ldots, l\}$, type is called an SD_j *agent*. Then there are altogether $l+1$ types of agents after taking into account the PD agents. Let n_{PD} denote the number of PD agents and n_{SD_j} the number of SD_j agents. Then the heterogeneity of the population is characterized by the *type-vector*

$$p \triangleq (n_{SD_1}, n_{SD_2}, \ldots, n_{SD_l}, n_{PD})$$

and obviously

$$p \in \mathcal{P}_n \triangleq \left\{ p \in \mathbb{Z}^{l+1} \,\middle|\, \sum_{i=1}^{l+1} p_i = n, p_i \geq 0 \right\}.$$

We find that after scaling up the agents' ratios by n, some comparisons involving the numbers of different types of agents can be easier to be made. For this reason, for an SD_j agent i, we define its *temper* to be

$$n^*_{SD_j} \triangleq n \frac{S_i - P_i}{T_i - R_i + S_i - P_i}. \tag{6.4}$$

The temper of a PD agent is defined to be zero. We study the effects of heterogeneity of the population when similar types are sufficiently apart in tempers; to be more precise, we assume

$$n^*_{SD_{i-1}} > \lceil n^*_{SD_i} \rceil \text{ and } \lfloor n^*_{SD_i} \rfloor > n^*_{SD_{i+1}}, \ \ \forall i \in \{2, \ldots, l-1\} \tag{6.5}$$

where for a real number a, $\lceil a \rceil$ and $\lfloor a \rfloor$ return the smallest integer greater than or equal to a and the largest integer less than or equal to a respectively. We use $n^C_{SD_i}$ to denote the number of cooperators among the SD_i players and n^C_{PD} that among the PD players. Then we stack all the $n^C_{SD_i}$ and n^C_{PD} together, and call the resulted $(l+1)$-dimensional vector the *distribution of cooperators* of the whole population

$$x \triangleq \left(n^C_{SD_1}, n^C_{SD_2}, \ldots, n^C_{SD_l}, n^C_{PD}\right).$$

The main goal of this chapter is to study given the type of a heterogeneous population $p \in \mathcal{P}$, how the system state x evolves over time t under the update rule (6.2) and the activation sequence of the agents. Here, we assume that the agents are activated *persistently* as follows.

Persistent activation assumption: For every agent $i \in \{1, \ldots, n\}$ and any time $t = 0, 1, 2, \ldots$, with probability one there exists some finite time $t' \geq t$ at which agent i is activated.

This is a very mild assumption that holds in most of the usual stochastic settings, e.g., when each agent is activated at a rate determined by a Poisson clock [31].

Given the asynchronous, nonlinear nature of the dynamics of $x(t)$, it is not clear whether $x(t)$ will converge at all. In fact, we will show that depending on the given type p, the state $x(t)$ almost surely converges to either a single state or a set of two states, and more importantly the convergence takes place globally in finite time. The key technical step is to construct a Lyapunov-like function. We provide the details of our main convergence results and their proofs in the next section.

6.3 Convergence of the best-response dynamics

To understand better the best response update rule (6.2), we first rewrite it in the form emphasizing the effects of the agents' tempers.

Lemma 6.1. *When an agent $i \in \{1, \dots, n\}$ updates its strategy $s_i(t)$ at time $t \geq 0$ according to (6.2), then*

$$s_i(t+1) = s^D \tag{6.6}$$

if i is a PD agent, and

$$s_i(t+1) = \begin{cases} s^C & n_C(t) < n^*_{SD_j} \\ s_i(t) & n_C(t) = n^*_{SD_j} \\ s^D & n_C(t) > n^*_{SD_j} \end{cases} \tag{6.7}$$

if i is an SD_j agent for some $j \in \{1, \dots, l\}$.

Proof. The update rule (6.2) dictates that agent i chooses the strategy that provides her the highest payoff against the population. Hence, according to the definition of u_i, agent i chooses the strategy corresponding to the bigger entry of $A_i s_C(t)$:

$$A_i s_C(t) = \begin{bmatrix} R_i n_C(t) + S_i(n - n_C(t)) \\ T_i n_C(t) + P_i(n - n_C(t)) \end{bmatrix}. \tag{6.8}$$

If agent i is a PD agent, from (6.3) it follows that $T_i > R_i > P_i > S_i$. So the second entry is always bigger, leading to (6.6). On the other hand, if agent i is an SD_j agent, using (6.8) to rewrite (6.2) gives

$$s_i(t+1) = \begin{cases} s^C & (R_i + P_i - T_i - S_i) n_C(t) > n(P_i - S_i) \\ s_i(t) & (R_i + P_i - T_i - S_i) n_C(t) = n(P_i - S_i) \\ s^D & (R_i + P_i - T_i - S_i) n_C(t) < n(P_i - S_i) \end{cases}.$$

From (6.3) we know that for an SD_j agent, it holds that

$$\left.\begin{matrix} T_i > R_i \\ S_i > P_i \end{matrix}\right\} \Rightarrow \begin{cases} R_i - T_i + P_i - S_i < 0 \\ P_i - S_i < 0 \end{cases}.$$

So

$$s_i(t+1) = \begin{cases} s^C & n_C(t) < n\frac{S_i - P_i}{T_i - R_i + S_i - P_i} \\ s_i(t) & n_C(t) = n\frac{S_i - P_i}{T_i - R_i + S_i - P_i} \\ s^D & n_C(t) > n\frac{S_i - P_i}{T_i - R_i + S_i - P_i} \end{cases},$$

which is the same as (6.7) because of (6.4). □

Lemma 6.1 tells us that a PD agent will always update to defect while an SD agent's decision depends on the comparison between its temper and the current total number of cooperators in the whole population. So the largest possible number of cooperators after every agent has updated at least once is $n - n_{PD} = \sum_{j=1}^{l} SD_j$. In this case, an immediate observation is that if

$$n - n_{PD} < n^*_{SD_l}, \tag{6.9}$$

and even when all the possible cooperators (namely all the SD players) are indeed cooperating, the least cooperative type of SD player, ie. SD_l players, will all cooperate. This motivates us to distinguish populations with various p's according to whether the inequality (6.9) holds.

Definition 6.1. *For any of those p with which* (6.9) *holds, we say the corresponding population is* biased *since in it even the least cooperative type of SD agents tends to cooperate; otherwise, we say the population is* unbiased.

We first observe that, from Lemma 6.1, if any PD agent is activated, her updated strategy is always to defect. Because of the persistent activation assumption, after a sufficient number of updates, all the PD agents have been activated at least once, then all of them stick to defection afterwards. We summarize it below.

Lemma 6.2. *Almost surely there exists a finite time t_{PD} such that*

$$x(t) = \left(n^C_{SD_1}(t), n^C_{SD_2}(t), \ldots, n^C_{SD_l}(t), 0\right) \quad \forall t \geq t_{PD}.$$

In the rest of the chapter, we use t_{PD} in Lemma 6.2 to denote the time that the convergence to the state stipulated in the lemma has taken place. The long-run behavior for biased populations is rather straightforward to establish, which is presented in the following theorem.

Theorem 6.1. *For a given biased population, almost surely there exists some time τ such that for all $t \geq \tau$*

$$x(t) = (n_{SD_1}, n_{SD_2}, \ldots, n_{SD_l}, 0).$$

Proof. For a biased population, it holds that

$$\left.\begin{array}{ll} n_C(t) \leq \sum_{j=1}^{l} n_{SD_j} & \forall t \geq t_{PD} \\ (6.9): \sum_{j=1}^{l} n_{SD_j} = n - n_{PD} < n^*_{SD_l} & \\ (6.5): n^*_{SD_l} \leq n^*_{SD_i} & \forall i \in \{1, \ldots, l\} \end{array}\right\} \Rightarrow n_C(t) < n^*_{SD_i} \qquad \forall i \in \{1, \ldots, l\}, \forall t \geq t_{PD}.$$

Hence, from Lemma 6.1, if any of the SD agents is activated after t_{PD}, she chooses cooperation as her strategy. Furthermore, since the agents are activated persistently, almost surely there exists some time $\tau \geq t_{PD}$ such that all of the SD agents have become activated before τ, and thus choose cooperation without changing it afterwards. So the proof is complete. □

However, the analysis of the long-run behavior for unbiased populations is much more complicated, and we need some additional technical notions. For an unbiased population, we can still examine the accumulated number of SD players according to their ordering, and more precisely we define the *benchmark*

$$k_p \triangleq \min \left\{ k \,\middle|\, k \in \{1, \ldots, l\}, \sum_{i=1}^{k} n_{SD_i} \geq n^*_{SD_k} \right\} \tag{6.10}$$

and call the SD_{k_p} agents the *benchmark agents* since such players, as will be shown later, indicate clearly who will always cooperate in the long run. The following property follows directly from this definition of k_p:

$$\sum_{i=1}^{j} n_{SD_i} < n^*_{SD_j} \qquad \forall j < k_p. \tag{6.11}$$

Now we are ready to present the first main result of this section, which identifies the invariant set of the system for unbiased populations. To do so, we define the following $(l+1)$-dimensional vectors

$$\begin{aligned}
x^* &\triangleq (n_{SD_1}, \ldots, n_{SD_{k_p-1}}, 0, \ldots, 0),\\
x^*_- &\triangleq (n_{SD_1}, \ldots, n_{SD_{k_p-1}}, \lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j}, 0, \ldots, 0),\\
x^*_+ &\triangleq (n_{SD_1}, \ldots, n_{SD_{k_p-1}}, \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}, 0, \ldots, 0).
\end{aligned}$$

Note that these three vectors differ only in their k_pth element; such elements in x^*_- and x^*_+ in general differ by one, and are the same if and only if $n^*_{SD_{k_p}}$ is an integer. For unbiased populations, we further classify them into two categories: we say the population is *clean-cut* if $\sum_{j=1}^{k_p-1} n_{SD_j} \geq n^*_{SD_{k_p}}$, otherwise, *ruffled*.

Proposition 6.1. *For a given unbiased population with the benchmark k_p, when the population is clean-cut, x^* is invariant; otherwise when it is ruffled, $\{x^*_-, x^*_+\}$ is.*

We need the following lemma in the proof.

Lemma 6.3. *If $x(t_0) \in \{x^*, x^*_-, x^*_+\}$ at some $t_0 \geq t_{PD}$, then*

$$n^C_{SD_i}(t_0+1) = n^C_{SD_i}(t_0) = 0 \qquad \forall i > k_p, \tag{6.12}$$

$$n^C_{SD_i}(t_0+1) = n^C_{SD_i}(t_0) = n_{SD_i} \qquad \forall i < k_p. \tag{6.13}$$

Proof. First we prove (6.12). At x^*, the number of cooperators in the population satisfies

$$n_C(t_0) = \sum_{j=1}^{k_p-1} n_{SD_j} \geq n^*_{SD_{k_p}} \geq \lfloor n^*_{SD_{k_p}} \rfloor. \tag{6.14}$$

Similarly, at both x^*_- and x^*_+, it holds that

$$n_C(t_0) \geq \sum_{j=1}^{k_p-1} n_{SD_j} + (\lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j}) = \lfloor n^*_{SD_{k_p}} \rfloor.$$

So at x^*, x^*_- and x^*_+, it is true that $n_C(t_0) \geq \lfloor n^*_{SD_{k_p}} \rfloor$, which in view of (6.5) implies that

$$n_C(t_0) > n^*_{SD_i} \quad \forall i > k_p.$$

So from Lemma 6.1 we know that if an SD_i, $i > k_p$, agent is active at t_0, her strategy at $t_0 + 1$ remains to defect, which proves (6.12).

Now we prove (6.13). At x^*,

$$n_C(t_0) = \sum_{j=1}^{k_p-1} n_{SD_j} \overset{(6.11)}{<} n_{SD_{k_p-1}},$$

while at both x^*_- and x^*_+,

$$n_C(t_0) \leq \sum_{j=1}^{k_p-1} n_{SD_j} + (\lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}) = \lceil n^*_{SD_{k_p}} \rceil$$
$$\overset{(6.5)}{\Longrightarrow} n_C(t_0) < n^*_{SD_{k_p-1}}.$$

So at x^*, x^*_- and x^*_+, $n_C(t_0) < n^*_{SD_{k_p-1}}$, which in view of (6.5) implies that

$$n_C(t_0) < n^*_{SD_i} \quad \forall i < k_p.$$

So from Lemma 6.1 we know that if an SD_i, $i < k_p$, agent is active at t_0, her strategy at $t_0 + 1$ remains to cooperate, which proves (6.13). □

Proof of Proposition 6.1. It suffices to show that for some $t_0 \geq t_{PD}$, in clean-cut populations

$$(x(t_0) = x^*) \;\Rightarrow\; (x(t) = x^* \quad \forall t \geq t_0)\,, \tag{6.15}$$

and in ruffled populations

$$\big(x(t_0) \in \{x^*_-, x^*_+\}\big) \;\Rightarrow\; \big(x(t) \in \{x^*_-, x^*_+\} \quad \forall t \geq t_0\big)\,. \tag{6.16}$$

We first observe that in a clean-cut population at x^*, $n^C_{SD_{k_p}}(t_0) = 0$. Then from (6.14) and Lemma 6.1, we know that

$$n^C_{SD_{k_p}}(t_0 + 1) = n^C_{SD_{k_p}}(t_0) = 0. \tag{6.17}$$

Combining (6.17) with (6.12) and (6.13) in Lemma 6.3, we conclude that (6.15) holds for $t = t_0 + 1$ and then by induction holds for any $t \geq t_0$.

We then observe that in a ruffled population at x^*_- and x^*_+,

$$n^C_{SD_{k_p}}(t_0) \in \big\{\lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j}, \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}\big\}. \tag{6.18}$$

Then one of the following three cases must takes place.

Case 1: $\lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j} = \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}$. Then $\lfloor n^*_{SD_{k_p}} \rfloor = \lceil n^*_{SD_{k_p}} \rceil = n^*_{SD_{k_p}}$, implying that $n_C(t_0) = n^*_{SD_{k_p}}$. Hence, from Lemma 6.1, if an SD_{k_p} agent is active at t_0, she will not change her strategy. Therefore,

$$n^C_{SD_{k_p}}(t_0 + 1) = n^C_{SD_{k_p}}(t_0) = \lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j} = \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}. \tag{6.19}$$

Case 2: $n^C_{SD_{k_p}}(t_0) = \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}$ and $\lceil n^*_{SD_{k_p}} \rceil \neq n^*_{SD_{k_p}}$. Then

$$n_C(t_0) = \lceil n^*_{SD_{k_p}} \rceil > n^*_{SD_{k_p}}.$$

Hence, from Lemma 6.1, if an SD_{k_p} agent is active at t_0, she updates to s^D at $t_0 + 1$. So

$$n^C_{SD_{k_p}}(t_0+1) = n^C_{SD_{k_p}}(t_0) - 1 = \lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j}. \tag{6.20}$$

Case 3: $n^C_{SD_{k_p}}(t_0) = \lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j}$ and $\lfloor n^*_{SD_{k_p}} \rfloor \neq n^*_{SD_{k_p}}$. Then

$$n_C(t_0) = \lfloor n^*_{SD_{k_p}} \rfloor < n^*_{SD_{k_p}}.$$

Hence, from 6.1, when an SD_{k_p} agent is active at t_0, she updates to s^C at $t_0 + 1$. So

$$n^C_{SD_{k_p}}(t_0+1) = n^C_{SD_{k_p}}(t_0) + 1 = \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}. \tag{6.21}$$

Moreover, when the active agent at t_0 is not an SD_{k_p} agent, we have that

$$n^C_{SD_{k_p}}(t_0+1) = n^C_{SD_{k_p}}(t_0). \tag{6.22}$$

Combining (6.19), (6.20), (6.21) and (6.22) together, we have

$$n^C_{SD_{k_p}}(t_0+1) \in \Big\{ \lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j}, \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j} \Big\}.$$

Combining the above deduction with (6.12) and (6.13) in Lemma 6.3, we conclude that (6.16) holds for $t = t_0 + 1$ and then by induction holds for any $t \geq t_0$. □

In fact, a much stronger statement than Proposition 6.1 can be made, which shows the invariant state/set is also globally attractive with probability one.

Theorem 6.2. *For a given unbiased population with the benchmark k_p, if the population is clean-cut, then almost surely there exists some time τ such that*

$$x(t) = x^* \qquad \forall t \geq \tau; \tag{6.23}$$

otherwise, almost surely there exists some time τ and two infinite time sequences $\{t^i_-\}_{i=1}^{\infty}$ and $\{t^i_+\}_{i=1}^{\infty}$ such that

$$x(t) \in \{x^*_-, x^*_+\} \qquad \forall t \geq \tau, \tag{6.24}$$

and

$$x(t^i_-) = x^*_-, \qquad x(t^i_+) = x^*_+ \qquad i = 1, 2, \ldots.$$

The theorem implies that almost surely after finite time the population reaches a state where every SD_i, $i = 1, \ldots, k_p - 1$, agent cooperates, and every SD_i, $i = k_p + 1, \ldots, l$, agent and every PD agent defects. In other words, almost surely all the agents that have a higher (resp. lower) temper than that of the benchmark agents, become cooperators (resp. defectors) after a sufficiently long, but finite, time. Moreover, if the population is clean-cut, all SD_{k_p} agents defect. Otherwise, if further $x^*_- = x^*_+$ or equivalently $\lfloor n^*_{SD_{k_p}} \rfloor = \lceil n^*_{SD_{k_p}} \rceil$, we have that $n^*_{SD_{k_p}} - \sum_{j=1}^{k_p-1} n_{SD_j}$ of the SD_{k_p} agents cooperate, and the rest defect; else in case $x^*_- \neq x^*_+$, the number of SD_{k_p} agents who cooperate fluctuates between $\lfloor n^*_{SD_{k_p}} \rfloor - \sum_{j=1}^{k_p-1} n_{SD_j}$ and $\lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}$.

To prove Theorem 6.2, we make use of a Lyapunov-like function $h(x)$ defined for populations with $k_p \geq 2$. Let $\alpha(x) = 0$ if $n^C_{SD_1} \neq n_{SD_1}$; otherwise, let $\alpha(x)$ return the largest index of those SD agents such that

$$n^C_{SD_j}(t) = n_{SD_j} \qquad \forall j \in \{1, \ldots, \alpha(x)\}, \tag{6.25}$$

$$\alpha(x) \leq k_p - 2. \tag{6.26}$$

Then $h(x)$ is defined to be

$$h(x) \triangleq \sum_{i=1}^{\alpha(x)+1} n^C_{SD_i}.$$

Obviously, $h(x)$ is lower bounded by zero and upper bounded by $\sum_{j=1}^{k_p-1} n_{SD_j}$. One main step in proving Theorem 6.2 is to show that h's upper bound is tight and along the system's trajectory $x(t)$, h almost surely reaches and stays at its maximum after finite time, after which all $SD_1, SD_2, \ldots, SD_{k_p-1}$ agents always cooperate. Then by using a similar function $g(x)$, we can prove all $SD_{k_p+1}, SD_{k_p+2}, \ldots, SD_l$ agents always defect after finite time, and the strategies of the SD_{k_p} players depends on whether the inequality $\sum_{j=1}^{k_p-1} n_{SD_j} \geq n^*_{SD_{k_p}}$ holds. Although along $x(t)$, h does not always increase, its lower bound can be tightened up as time grows. To be more specific, we will construct an infinite set $\mathcal{B}$ of time instants t_b, such that for any $t_b \in \mathcal{B}$, it holds that $h(t) \geq h(t_b)$ for all $t \geq t_b$; more precisely

$$\mathcal{B} = \{t_b\} \triangleq \left\{ t \geq t_{PD} \,|\, n_C(t) = \sum_{j=1}^{k_p-1} n_{SD_j} \text{ if clean-cut or } n_C(t) \in \{\lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil\} \text{ otherwise} \right\}.$$

Then at t_b, the number of cooperators in a clean-cut population is x^*, and that in a ruffled population is x^*_- or x^*_+. Now we show by the following two lemmas that such a constructed $\mathcal{B}$ is not only non-empty but infinite.

Lemma 6.4. *In a clean-cut population, if at some time $T > t_{PD}$,*

$$n_C(T) < \sum_{j=1}^{k_p-1} n_{SD_j}, \tag{6.27}$$

then almost surely there exists a finite time $T' > T$ such that

$$n_C(T') = \sum_{j=1}^{k_p-1} n_{SD_j}. \tag{6.28}$$

In a ruffled population, if at some time $T > t_{PD}$,

$$n_C(T) < \lfloor n^*_{SD_{k_p}} \rfloor, \tag{6.29}$$

then almost surely there exists some finite time T' such that

$$n_C(T') \in \left\{ \lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil \right\}. \tag{6.30}$$

Proof. For the clean-cut population, we first prove by contradiction that there exists some finite time $\bar{T} > T$ such that

$$n_C(\bar{T}) \geq \sum_{j=1}^{k_p-1} n_{SD_j}. \tag{6.31}$$

Assume the contrary, that is

$$n_C(t) < \sum_{j=1}^{k_p-1} n_{SD_j} \qquad \forall t > T. \tag{6.32}$$

Then from (6.11), for all $t > T$ we have that

$$n_C(t) < n^*_{SD_{k_p-1}} \overset{(6.5)}{\Longrightarrow} n_C(t) < n^*_{SD_i} \quad \forall i < k_p.$$

Hence, from Lemma 6.1 and the persistent activation assumption, we know that almost surely there exists a finite time τ, before which all of the SD_i, $i < k_p$, agents, have updated at least once to fix their strategies to cooperation. Then

$$x(\tau) = \left(n_{SD_1}, \ldots, n_{SD_{k_p-1}}, n^C_{SD_{k_p}}(\tau), \ldots, n^C_{SD_l}(\tau), 0 \right),$$

where the zero component follows from Lemma 6.2, implying

$$n_C(\tau) = \sum_{j=1}^{l} n_{SD_i}(\tau) \geq \sum_{j=1}^{k_p-1} n_{SD_i},$$

which contradicts (6.32). So (6.31) must be true.

Further, since n_C takes integer values and changes at most by one each time, the transition of n_C from (6.27) to (6.31) implies the existence of T' to make n_C become $\sum_{j=1}^{k_p-1} n_{SD_j}$ at T', and thus (6.28) is proved.

For the ruffled population, similarly, we first prove by contradiction the existence of a finite time $\bar{T} > T$ such that

$$n_C(\bar{T}) \geq \lfloor n^*_{SD_{k_p}} \rfloor. \tag{6.33}$$

Assume the contrary, that is

$$n_C(t) < \lfloor n^*_{SD_{k_p}} \rfloor \qquad \forall t > T. \tag{6.34}$$

From $\lfloor n^*_{SD_{k_p}} \rfloor \leq n^*_{SD_{k_p}}$, it follows that for all $t > T$,

$$n_C(t) < n^*_{SD_{k_p}} \overset{(6.5)}{\Longrightarrow} n_C(t) < n^*_{SD_i} \quad \forall i \leq k_p.$$

Then from Lemma 6.1, the persistent activation assumption and Lemma 6.2, almost surely there is a finite time $\tau > T$ such that

$$x(\tau) = \left(n_{SD_1}, \ldots, n_{SD_{k_p}}, n^C_{SD_{k_p+1}}(\tau), \ldots, n^C_{SD_l}(\tau), 0\right).$$

Then because of the definition of k_p,

$$n_C(\tau) = \sum_{j=1}^{l} n^C_{SD_i}(\tau) \geq \sum_{j=1}^{k_p} n_{SD_i} \geq n^*_{SD_{k_p}} \geq \lfloor n^*_{SD_{k_p}} \rfloor,$$

which contradicts (6.34). So (6.33) is true and thus the transition of n_C from (6.29) to (6.33) implies (6.30). □

Lemma 6.5. *In a clean-cut population, if at $T > t_{PD}$,*

$$n_C(T) > \sum_{j=1}^{k_p-1} n_{SD_j}, \tag{6.35}$$

then almost surely there exists a finite time $T' > T$ such that

$$n_C(T') = \sum_{j=1}^{k_p-1} n_{SD_j}. \tag{6.36}$$

In a ruffled population, if at $T > t_{PD}$,

$$n_C(t) > \lceil n^*_{SD_{k_p}} \rceil, \tag{6.37}$$

then almost surely there exists a finite time T' such that

$$n_C(T') \in \left\{ \lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil \right\}. \tag{6.38}$$

Proof. This lemma can be proven by contradiction following the analogous steps as those in the proof for Lemma 6.4. □

Since Lemma 6.4 covers the situation for $t > t_{PD}$, $n_C(t) < \sum_{j=1}^{k_p-1} n_{SD_j}$, Lemma 6.5 for the situation $n_C(t)$ greater than the sum, and the situation n_C equals the sum corresponds to $t \in \mathcal{B}$, we have shown that for any $t > t_{PD}$, either $t \in \mathcal{B}$ or there is a finite $t' > t$ satisfying $t' \in \mathcal{B}$. So we have actually proven the following.

Lemma 6.6. *The set $\mathcal{B}$ is infinite.*

After knowing $\mathcal{B}$ has infinite time instants in it, we show the nondecreasing property of $h(x)$ along $x(t)$ at $t \in \mathcal{B}$.

Lemma 6.7. *If $t_b \in \mathcal{B}$, then*

$$h(t) \geq h(t_b) \qquad \forall t \geq t_b. \tag{6.39}$$

We need the following two lemmas in the proof.

Lemma 6.8. *For all $t_b \in \mathcal{B}$, it holds that*

$$n_C(t_b) < n^*_{SD_{k_p-1}}. \tag{6.40}$$

Proof. When the population is clean-cut,

$$n_C(t_b) = \sum_{j=1}^{k_p-1} n_{SD_j} \overset{(6.11)}{\Longrightarrow} n_C(t_b) < n^*_{SD_{k_p-1}}.$$

Otherwise,

$$\left.\begin{array}{c} n_C(t_b) \in \left\{\lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil\right\} \Rightarrow n_C(t_b) \le \lceil n^*_{SD_{k_p}} \rceil \\ (6.5): \quad \lceil n^*_{SD_k} \rceil < n^*_{SD_{k_p-1}} \end{array}\right\} \Rightarrow n_C(t_b) < n^*_{SD_{k_p-1}}.$$

Hence, (6.40) holds for all unbiased populations. □

Lemma 6.9. *If $t_b \in \mathcal{B}$, then*

$$n_C(t) < n^*_{SD_{\alpha(t_b)+1}} \qquad \forall t \ge t_b. \tag{6.41}$$

Proof. We prove by contradiction. Assume on the contrary that (6.9) can be violated. Then let $t_1 \ge t_b + 1$ be the first such violation time. So

$$n_C(t_1) \ge n^*_{SD_{\alpha(t_b)+1}} \Rightarrow n_C(t_1) \ge \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil. \tag{6.42}$$

However, from Lemma 6.8 we know that

$$n_C(t_b) < n^*_{SD_{k_p-1}},$$

and since (6.26) is in force for all α including $\alpha = 0$, in view of (6.5), we obtain

$$n_C(t_b) < n^*_{SD_{\alpha(t_b)+1}}. \tag{6.43}$$

By comparing $n_C(t_b)$ and $n_C(t_1)$ from (6.43) and (6.42) and knowing that n_C changes at most by one per time, we obtain

$$\exists t' \in [t_b, t_1 - 1]: \quad n_C(t') = \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil - 1. \tag{6.44}$$

Let t_s be the greatest of all such possible t'. Then in view of (6.42),

$$n_C(t) \ge \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil \qquad \forall t \in [t_s + 1, t_1]. \tag{6.45}$$

Now we show that if an agent changes her strategy from D to C within $[t_s + 1, t_1]$, it must be an $SD_{\alpha(t_b)+1}$ agent. Let t_c denote some time in $[t_s + 1, t_1]$. For an SD_j, $j > \alpha(t_b) + 1$, agent, it follows that

$$\left.\begin{array}{l} (6.45) \Rightarrow n_C(t_c - 1) \ge \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil - 1 \\ j > \alpha(t_b) + 1 \overset{(6.5)}{\Longrightarrow} \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil - 1 \ge n^*_{SD_j} \end{array}\right\} \Rightarrow n_C(t_c - 1) \ge n^*_{SD_j}, \tag{6.46}$$

which, from Lemma 6.1, implies that the SD_j cannot change her choice to C at t_c. On the other hand, for an agent SD_j, $j < \alpha(t_b) + 1$, we have

$$j \leq \alpha(t_b) \xRightarrow{(6.25)} n^C_{SD_j}(t_b) = n_{SD_j}, \tag{6.47}$$

and by the definition of t_1, since $t_c - 1 \leq t_1$,

$$n^C_{SD_j}(t_c - 1) = n^C_{SD_j}(t_b), \tag{6.48}$$

which together imply that

$$n^C_{SD_j}(t_c - 1) = n_{SD_j} \qquad \forall t_c \in [t_s + 1, t_1]. \tag{6.49}$$

Therefore, all SD_j agents are already cooperators at $t_c - 1$ and hence do not change their choices to C at t_c. So only an $SD_{\alpha(t_b)+1}$ agent can change her choice from D to C within $[t_s + 1, t_1]$.

Now if $t_1 > t_s + 1$, from Lemma 6.1, (6.45) implies that no SD_i agent changes her choice to C during $[t_s + 2, t_1]$. Hence, no agent changes her choice from D to C during $[t_s + 2, t_1]$. So

$$n_C(t) \leq n_C(t_s + 1) \qquad \forall t \in [t_s + 1, t_1]. \tag{6.50}$$

On the other hand, (6.50) also holds if $t_1 = t_s + 1$, verifying it in general. Now, (6.45) and the equality $n_C(t_s) = \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil - 1$ imply

$$n_C(t_s + 1) = \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil. \tag{6.51}$$

Then (6.45), (6.50) and (6.51) lead to

$$n_C(t) = \lceil n^*_{SD_{\alpha(t_b)+1}} \rceil \qquad \forall t \in [t_s + 1, t_1]. \tag{6.52}$$

So

$$x(t_s + 1) = x(t_1)$$

implying that

$$n_C(t_s) = n_C(t_1) \geq n^*_{SD_{\alpha(t_b)+1}},$$

which is a contradiction since $t_s < t_1$ and t_1 was defined as the first time after t_b at which n_C becomes non-less than $n^*_{SD_{\alpha(t_b)+1}}$. □

Proof of Lemma 6.7. We prove by contradiction. Assume on the contrary that (6.39) can be violated at some time $t_1 \geq t_b + 1$. Hence, for some $i = 1, \ldots, n_{SD_{\alpha(t_b)+1}}$, an SD_i agent has changed her choice from C to D at t_1. So from Lemma 6.1,

$$n_C(t_1 - 1) > n^*_{SD_i}. \tag{6.53}$$

On the other hand, it follows from (6.5) that $n^*_{SD_i} \geq n^*_{SD_{\alpha(t_b)+1}}$. Hence, (6.53) implies

$$n_C(t_1 - 1) > n^*_{SD_{\alpha(t_b)+1}},$$

which contradicts Lemma 6.9 since $t_1 - 1 \geq t_b$. □

Lemma 6.7 has shown how the lower bound of h is tightened up as time grows. In the following two lemmas, we show that h indeed reaches its maximum. First, we observe that the following statement follows directly from Lemma 6.9.

Corollary 6.1. *For a fixed $t_b \in \mathcal{B}$, consider the first time after t_b that an $SD_{\alpha(t_b)+1}$ agent, whose strategy was s^D at t_b, is activated. Then this agent updates her strategy to s^C, and does not change it afterwards.*

Then we prove that h reaches its maximum.

Lemma 6.10. *Almost surely there exists some time t_h, at which h reaches and after which remains at its maximum, i.e., $h(t) = \sum_{j=1}^{k_p-1} n_{SD_j}$ for all $t \geq t_h$.*

Proof. If there exists $t_b \in \mathcal{B}$ such that $h(t_b) = \sum_{j=1}^{k_p-1} n_{SD_j}$, then from Lemma 6.7, $h(t) \geq \sum_{j=1}^{k_p-1} n_{SD_j}$ for all $t \geq t_b$, and thus the result holds straightforwardly. Otherwise, the only other possibility is that for each $t_b \in \mathcal{B}$, $h(t_b) < \sum_{j=1}^{k_p-1} n_{SD_j}$. We will show below that this, however, will never happen. In this case, for every $t_b \in \mathcal{B}$, there exists an $SD_{\alpha(t_b)+1}$ agent whose strategy is s^D at t_b. On the other hand, due to the persistent activation assumption, every such agent almost surely will become activated at some finite future time, and in view of Corollary 6.1, will update her choice to C and will not change it afterwards. This cannot be true since $\mathcal{B}$ is infinite according to Lemma 6.6, but the number of $SD_{\alpha(t_b)+1}$ agents are finite. So the contradiction completes the proof. □

For the same reason of defining h, in order to show that all of the $SD_{k_p+1}, \ldots, SD_l$ agents eventually become defectors, we define another Lyapunov-like function $\gamma(x)$ defined for populations with $k_p \leq l-1$. Let $\gamma(x) = 0$ if $n^C_{SD_l} \neq 0$; otherwise, let $\gamma(x)$ return the smallest index j of those SD_j agents such that

$$n^C_{SD_j}(t) = 0 \qquad \forall j \in \{\gamma(t), \ldots, l\},$$
$$\gamma(t) \geq k_p + 2.$$

Clearly g has a minimum of 0 and indeed similar to the property of h, one can show that g eventually reaches its minimum.

Lemma 6.11. *Almost surely there exists some time t_g at which g reaches and remains at its minimum, i.e., $g(t) = 0$ for all $t \geq t_h$.*

Now we are ready to prove the main result.

Proof of Theorem 6.2. If $k_p \geq 2$, then in view of Lemma 6.10 and according to the definition of h, there exists some time t_h such that

$$n^C_{SD_j}(t) = n_{SD_j} \qquad \forall t \geq t_h,\ \forall j \in \{1 \ldots, k_p - 1\}. \tag{6.54}$$

On the other hand, if $k_p \leq l-1$, then in view of Lemma 6.11 and according to the definition of g, there exists some time t_g such that

$$n^C_{SD_j}(t) = 0 \qquad \forall t \geq t_g,\ \forall j \in \{k_p + 1, \ldots, l\}. \tag{6.55}$$

Let $t_m = \max\{t_h, t_g\}$. From (6.54) and (6.55), it follows

$$x(t) = \left(n_{SD_1}, \ldots, n_{SD_{k_p-1}}, n^C_{SD_{k_p}}(t), 0, \ldots, 0\right) \quad \forall t \geq t_m. \tag{6.56}$$

Clearly (6.56) also holds if $k_p = 1$ or $k_p = l$, implying that it holds in general for any unbiased population. Now Consider $t_b \in \mathcal{B}, t_b \geq t_m$. If the population is clean-cut, then by the definition of $\mathcal{B}$,

$$n_C(t_b) = \sum_{j=1}^{k_p-1} n_{SD_j} \xrightarrow{(6.56)} n^C_{SD_{k_p}}(t_b) = n_C(t_b) - \sum_{j=1}^{k_p-1} n_{SD_j} = 0.$$

Hence, $x(t_b) = x^*$, and in view of Lemma 6.1, $x(t) = x^*$ for all $t \geq t_b$. On the other hand, if the population is ruffled, then

$$\left.\begin{array}{c} n_C(t_b) \in \left\{\lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil\right\} \\ (6.56): n^C_{SD_{k_p}}(t_b) = n_C(t_b) - \sum_{j=1}^{k_p-1} n_{SD_j} \end{array}\right\} \Rightarrow$$

$$n^C_{SD_{k_p}}(t_b) \in \left\{\lfloor n^*_{SD_{k_p}} \rfloor - \textstyle\sum_{j=1}^{k_p-1} n_{SD_j}, \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}\right\}.$$

Hence, $x(t_b) \in \{x^*_-, x^*_+\}$, and in view of Lemma 6.1, $x(t) \in \{x^*_-, x^*_+\}$ for all $t \geq t_b$. Therefore, (6.23) and (6.24) are proven by taking $\tau = t_b$.

To construct the two infinite subsequences, if $x^*_- = x^*_+$, the result is trivial. Otherwise, let $\tau_i > \tau$ be the ith time after $t = \tau$ that an SD_{k_p} agent is activated. The time sequence $\{\tau_i\}$ is almost surely infinite since the agents are activated persistently. From (6.24), $x(\tau_i) \in \left\{x^*_-, x^*_+\right\}$. Without loss of generality assume $x(\tau_1) = x^*_-$. From the definition of x^*_-,

$$n_C(\tau_1) = \sum_{j=1}^{k_p-1} n_{SD_j} + n^C_{SD_{k_p}}(\tau_1) = \lfloor n^*_{SD_{k_p}} \rfloor. \tag{6.57}$$

It can be easily verified that $x(t)$ does not change within $[\tau_1, \tau_2 - 1]$. Hence, $n_C(\tau_2 - 1) = n_C(\tau_1)$. On the other hand, $x^*_- \neq x^*_+$ yields $\lfloor n^*_{SD_{k_p}} \rfloor < n^*_{SD_{k_p}} < \lceil n^*_{SD_{k_p}} \rceil$. Hence, because of (6.57), $n_C(\tau_2 - 1) < n^*_{SD_{k_p}}$, which in view of Lemma 6.1 implies that the active SD_{k_p} agent at τ_2 switches her choice from D to C. Hence, $n^C_{SD_{k_p}}(\tau_2) = n^C_{SD_{k_p}}(\tau_1) + 1 = \lceil n^*_{SD_{k_p}} \rceil - \sum_{j=1}^{k_p-1} n_{SD_j}$. Therefore, $x(\tau_2) = x^*_+$. Similarly, it can be shown that $x(\tau_3) = x^*_-$. In general, by induction

$$x(\tau_{2r+1}) = x^*_-, \qquad x(\tau_{2r+2}) = x^*_+ \qquad \forall r = 0, 1, 2, \ldots.$$

Taking $\{t^i_-\}_{i=1}^{\infty} = \{\tau_{2r+1}\}_{r=0}^{\infty}$ and $\{t^i_+\}_{i=1}^{\infty} = \{\tau_{2r}\}_{r=0}^{\infty}$, we arrive at the conclusion. □

Remark 6.1. *Theorem 6.2 also holds when condition* (6.5) *is relaxed to the following*

$$n^*_{SD_i} > n^*_{SD_{i+1}} \qquad \forall i \in \{1, \ldots, l-1\};$$

however, then the agents must be activated pairwise persistently*, that is for any pair of agents i and j and each time t, with probability one there exists some finite time $t' > t$ such that i and*

j are activated consecutively at t' and $t'+1$, respectively. Although stronger than the persistent activation assumption, the pairwise persistent activation assumption is satisfied in most stochastic settings as well, particularly when agents are activated independently, e.g., according to Poisson clocks.

The following result can be derived directly from Theorems 6.1 and 6.2, and specifies the number of long-run cooperators.

Corollary 6.2. *For a population with $p \in \mathcal{P}_n$, almost surely there exists some time τ such that for all $t \geq \tau$, if the population is biased, $n_C(t) = \sum_{i=1}^{l} n_{SD_i}$; otherwise, if it is clean-cut,*

$$n_C(t) = \sum\nolimits_{i=1}^{k_p-1} n_{SD_i};$$

and if it is ruffled,

$$n_C(t) \in \left\{ \lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil \right\},$$

with two infinite time sequences $\{t^i_-\}_{i=1}^{\infty}$ and $\{t^i_+\}_{i=1}^{\infty}$ satisfying for $i = 1, 2, \ldots$,

$$n_C(t^i_-) = \lfloor n^*_{SD_{k_p}} \rfloor, \qquad n_C(t^i_+) = \lceil n^*_{SD_{k_p}} \rceil.$$

Corollary 6.2 and Lemma 6.8 can be used the derive lower and upper bounds for the number of long-run cooperators.

Corollary 6.3. *For an unbiased population with benchmark k_p, almost surely there exists some time τ such that for all $t \geq \tau$,*

$$n^*_{SD_{k_p}} \leq n_C(t) < n^*_{SD_{k_{p-1}}}.$$

After having presented the mathematical analysis of the long-run behavior of the decision-making dynamics, in the next section, we look into how the number of long-run cooperators changes when the cooperation tendencies of some SD players are manipulated.

6.4 The level-off phenomenon

For heterogeneous populations, important research questions arise that do not show up for homogeneous populations. In particular, we are interested in knowing whether increasing some individuals' tendencies to cooperate results in the rise of the cooperation level of the whole population. Mainly through simulations, we have provided a negative answer to this question in [139]. In this chapter, we give more thorough theoretical analysis showing that the total number of cooperators in the long run, denoted by n^f_C, may initially increase as the tendencies of a portion of individuals increases, but levels off as the portion grows until the size of the portion passes a threshold, after which the number of cooperators may continue to grow. We call this particular relationship between the number of long-run cooperators and the portion of manipulated population the *level-off phenomenon*. Let us first illustrate this phenomenon by an example.

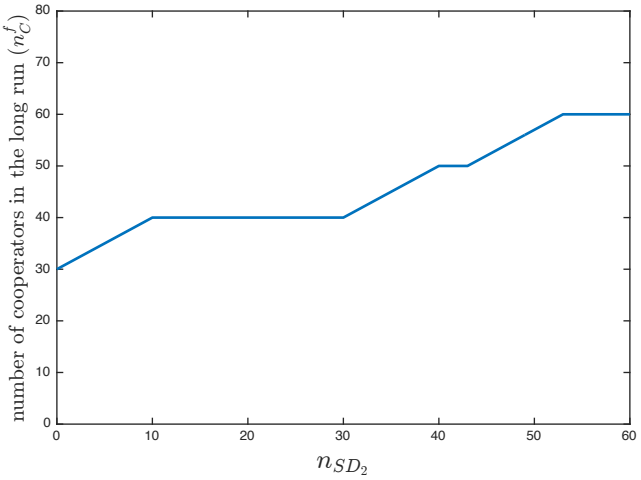


Figure 6.1: For Example 1, the number of long-run cooperators n_C^f against the number of SD_5 agents r, whose types have changed to SD_2.

Example 6.1. *Consider the decision-making dynamics formulated in Sec. 6.2 for a population of 100 agents with six different types, namely* $n = 100$ *and* $l = 5$. *Set the tempers to be*

$$(n^*_{SD_1}, n^*_{SD_2}, n^*_{SD_3}, n^*_{SD_4}, n^*_{SD_5}) = (75, 60, 50, 40, 15),$$

and the type-vector

$$p = (7, 0, 3, 20, 60, 10).$$

For $r = 0, 1, \ldots 60$ *agents of the* SD_5 *type, we change them to* SD_2 *agents. We plot* n_C^f *against* r *in Figure 6.1. Clearly, as the number of* SD_2 *agents increases, the number of long-run cooperators* n_C^f *first increases as* r *grows from 0 to 10, levels off for* $10 \leq r \leq 30$, *increases again for* $30 \leq r \leq 40$, *and experiences two more level-offs afterwards. So increasing* r *does not necessarily increase the number of long-run cooperators.*

To explain such level-off phenomena mathematically, we first observe from Figure 6.1 that every plateau in the curve takes the value of one of the integer-valued tempers $n^*_{SD_b}, b \in \{1, \ldots, l\}$. We formulate this observation more rigorously in the following proposition.

Proposition 6.2. *For* $b \in \{1, \ldots, l\}$, *assume* $n^*_{SD_b}$ *is an integer. Then almost surely* $n_C^f = n^*_{SD_b}$, *if and only if* $\sum_{s=1}^{b-1} p_s \leq n^*_{SD_b} \leq \sum_{s=1}^{b} p_s$.

Proof. (sufficiency) Since

$$\left.\begin{aligned} \sum_{s=1}^{b-1} p_s \leq n^*_{SD_b} &\overset{(6.5)}{<} n^*_{SD_{b-1}} \\ \sum_{s=1}^{b} p_s \geq n^*_{SD_b} & \end{aligned}\right\} \Rightarrow k_p = b,$$

it follows from Corollary 6.2 and $\sum_{s=1}^{b-1} p_s \leq n^*_{SD_b}$ that almost surely $n^f_C(p) = n^*_{SD_b}$.
(necessity) Let $n^f_C(p) = n^*_{SD_b}$. First $n^*_{SD_b} \leq \sum_{s=1}^{b} p_s$ is proven by contradiction. Assume the contrary, i.e., $n^*_{SD_b} > \sum_{s=1}^{b} p_s$, then $k_p \geq b+1$. Consider the situation when the population is clean-cut, i.e., $\sum_{s=1}^{k_p-1} p_s \geq n^*_{SD_{k_p}}$. Then Corollary 6.2 implies $n^f_C(p) = \sum_{s=1}^{k_p-1} p_s$. On the other hand, by the definition of k_p, $\sum_{s=1}^{k_p-1} p_s < n^*_{SD_{k_p-1}}$. Hence, $n^f_C(p) < n^*_{SD_{k_p-1}}$. So in view of (6.5), $n^f_C(p) < n^*_{SD_b}$, which is impossible. Therefore, the population is ruffled, i.e., $\sum_{s=1}^{k_p-1} p_s < n^*_{SD_{k_p}}$. Hence, in view of Corollary 6.2, $n^f_C(p) = n^*_{SD_b} = n^*_{SD_{k_p}}$, but this is in contradiction with $k_p \geq b+1$. So $n^*_{SD_b} \leq \sum_{s=1}^{b} p_s$ must be true.

Now $\sum_{s=1}^{b-1} p_s \leq n^*_{SD_b}$ is proven by contradiction. Assume the contrary, i.e., $\sum_{s=1}^{b-1} p_s > n^*_{SD_b}$. If $\sum_{s=1}^{b-1} p_s < n^*_{SD_{b-1}}$, then $k_p = b$, and in view of Corollary 6.2, $n^f_C(p) = \sum_{s=1}^{b-1} p_s > n^*_{SD_b}$, which is impossible. Hence, $\sum_{s=1}^{b-1} p_s \geq n^*_{SD_{b-1}}$, yielding $k_p \leq b-1$. On the other hand, in view of Corollary 6.3, $n^f_C(p) \geq n^*_{SD_{k_p}}$, which according to (6.5) and $k_p \leq b-1$ results in $n^f_C(p) > n^*_{SD_b}$, a contradiction. Hence, $\sum_{s=1}^{b-1} p_s \leq n^*_{SD_b}$. □

The proposition shows the existence of a range of type-vectors that almost surely will lead to $n^*_{SD_b}$ cooperators in the population in the long run. The following theorem takes advantage of this result, and determines precisely those changes in the types of the agents in order to achieve and maintain $n^*_{SD_b}$ cooperators in the long run. For a type-vector p, denote by $p^r_{i\to j}$, $i,j \in \{1,\ldots,l\}$, the new type-vector after the type of $r \geq 0$ agents changes from SD_i to SD_j. When all the tempers $n^*_{SD_b}$ are integers, from Corollary 6.2, we know that $n^f_C(p)$ always exists for any $p \in \mathcal{P}_n$.

Theorem 6.3 (sustainable cooperation levels)**.** *For a given population with $p \in \mathcal{P}_n$, assume all the tempers $n^*_{SD_b}$ are integers and assume there exists some $b \in \{1,\ldots,l\}$ such that $n^f_C(p) = n^*_{SD_b}$. Then*

1. *$n^f_C(p^r_{i\to j}) > n^f_C(p)$, if and only if $i \geq b, j \leq b-1$ and $r > n^*_{SD_b} - \sum_{s=1}^{b-1} p_s$,*
2. *$n^f_C(p^r_{i\to j}) < n^f_C(p)$, if and only if $i \leq b, j \geq b+1$ and $r > \sum_{s=1}^{b} p_s - n^*_{SD_b}$,*
3. *$n^f_C(p^r_{i\to j}) = n^f_C(p)$, otherwise.*

Proof. Using Proposition 6.2, it can be shown that $n^f_C(p^r_{i\to j}) = n^f_C(p)$, if and only if one of the following holds

1. $i = j$,
2. $i \geq b+1$ and $j \geq b$,
3. $i \geq b+1, j \leq b-1$ and $r \leq n^*_{SD_b} - \sum_{s=1}^{b-1} p_s$,
4. $i = b, j \leq b-1$ and $r \leq n^*_{SD_b} - \sum_{s=1}^{b-1}$,
5. $i = b, j \geq b+1$ and $r \leq \sum_{s=1}^{b} -n^*_{SD_b}$,
6. $i \leq b-1$ and $j \leq b$,

7. $i \leq b-1, j \geq b+1$ and $r \leq \sum_{s=1}^{b} p_s - n^*_{SD_b}$.

Hence, in order to have $n^f_C(p^r_{i\to j})$ greater than (resp. less than) $n^f_C(p)$, all of the above cases must be violated. This results in either $i \geq b, j \leq b-1$ and $r > n^*_{SD_b} - \sum_{s=1}^{b-1} p_s$ or $i \leq b, j \geq b+1$ and $r > \sum_{s=1}^{b} p_s - n^*_{SD_b}$. It can be verified that only in the first case, $n^f_C(p^r_{i\to j})$ becomes greater than $n^f_C(p)$, and only in the second case, $n^f_C(p^r_{i\to j})$ becomes less than $n^f_C(p)$. Hence, in any other case, $n^f_C(p^r_{i\to j}) = n^f_C(p)$, which completes the proof. □

Theorem 6.3 confirms the existence of cooperation levels that are robust against changes in the cooperation tendencies of the agents. Namely, unless one of the first two cases in the theorem takes place, the number of long-run cooperators is robust against the changes in types of the population. Note that this robustness is against both increasing and decreasing cooperation tendencies.

Another usage of Theorem 6.3 is to determine quantitatively the widths of the plateaus for curves like Figure 6.1. In Example 6.1, since $i \geq b$ and $j \leq b-1$, Case 1) of Theorem 6.3 implies that increasing the type of r SD_5 agents to SD_2, does not increase the total number of cooperators if and only if $r \leq n^*_{SD_4} - \sum_{s=1}^{3} p_s = 40 - 20 = 20$. Therefore, the width of the first plateau is 20.

The following proposition discusses what happens when the number of cooperators is not equal to any temper $n^*_{SD_b}$.

Proposition 6.3. *Given a population with $p \in \mathcal{P}_n$, if the number of long-run cooperators is some constant between two consecutive tempers of the agents, $n^*_{SD_b}$ and $n^*_{SD_{b-1}}$, $b \in \{2, \ldots, l\}$, where $n^*_{SD_{b-1}}$ is an integer, then changing an SD_i agent to SD_j where $j < b \leq i$, increases the number of long-run cooperators, i.e.,*

$$n^*_{SD_b} < n^f_C(p) < n^*_{SD_{b-1}} \Rightarrow n^f_C(p^1_{i\to j}) > n^f_C(p),\ j < b \leq i.$$

Proof. From Corollary 6.2, the condition $n^*_{SD_b} < n^f_C(p) < n^*_{SD_{b-1}}$ implies $n^f_C(p) = \sum_{s=1}^{k_p-1} p_s \geq n^*_{SD_{k_p}}$. Then, since $\sum_{s=1}^{k_p-1} p_s < n^*_{SD_{k_p}-1}$, it holds that $k_p = b$. Let $\hat{p} = p^1_{i\to j}$. Clearly $\sum_{s=1}^{b-1} \hat{p}_s = \sum_{s=1}^{k_p-1} \hat{p}_s = \sum_{s=1}^{k_p-1} p_s + 1$. Hence,

$$n^*_{SD_b} < \sum_{s=1}^{b-1} \hat{p}_s \leq n^*_{SD_{b-1}}.$$

If $\sum_{s=1}^{b-1} \hat{p}_s < n^*_{SD_{b-1}}$, then $k_{\hat{p}} = b$ and the population is clean-cut. So in view of Corollary 6.2, $n^f_C(\hat{p}) = \sum_{s=1}^{b-1} \hat{p}_s = \sum_{s=1}^{k_p-1} p_s + 1 = n^f_C(p) + 1$. If on the other hand, $\sum_{s=1}^{b-1} \hat{p}_s = n^*_{SD_{b-1}}$, then it can be shown that $k_{\hat{p}} = b-1$. Hence, from Corollary 6.3 and since $n^*_{SD_{b-1}}$ is an integer, we have that $n^f_C(\hat{p}) = n^*_{SD_{b-1}} > n^f_C(p)$. □

As shown in the proof of Proposition 6.3, the condition $n^*_{SD_b} < n^f_C(p) < n^*_{SD_{b-1}}$ implies $k_p = b$ and $\sum_{s=1}^{k_p-1} p_s \geq n^*_{SD_{k_p}}$. Hence, from Theorem 6.2, the condition $j < b \leq i$ in Proposition 6.3 implies that a defector changes to a cooperator. On the other hand, in view of Theorem 6.3, the above proposition implies that sustainable cooperation levels only maintain at the tempers of the agents. Therefore, Proposition 6.3 together with Theorem 6.3 provide a complete characterization of the level-off phenomena.

6.5 Controlling the number of cooperators

Now that we know the behavior of the population game after a sufficient number of time steps, we are interested in investigating whether it is possible to control the number of cooperators in the final state. Before proceeding to the rest of this section, we present the following reformulation of Corollary 6.2 that we use during the proofs. Given $p \in \mathcal{P}_n$, we denote the number of cooperators in the population at time t by $n_C(p,t)$. We also extend the definition of k_p as follows. For the temper n^* and type vector $p \in \mathcal{P}_n$, define

$$k_p \triangleq \min \left\{ k \,\middle|\, k \in \{1,\dots,l\}, \sum_{i=1}^{k} p_i \geq n^*_{SD_k} \right\}$$

if $\sum_{i=1}^{l} p_i \geq n^*_{SD_l}$ and $k_p = l+1$ otherwise. Then the total number of cooperators in the long run can be determined as follows.

Corollary 6.4 (reformulation of Corollary 6.2)**.** *For a population with the type vector $p \in \mathcal{P}_n$ and temper n^*, almost surely there exists some time τ such that if $k_p = l+1$ holds, or $k_p \neq l+1$ and $\sum_{i=1}^{k_p-1} p_i \geq n^*_{SD_{k_p}}$ hold, then*

$$\exists \tau : \quad n_C(p,t) = \sum_{i=1}^{k_p-1} p_i \quad \forall t \geq \tau.$$

and if $k_p \neq l+1$ and $\sum_{i=1}^{k_p-1} p_i < n^*_{SD_{k_p}}$ holds, then

$$\exists \tau : \quad n_C(p,t) \in \left\{ \lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil \right\} \quad \forall t \geq \tau$$

with two infinite time sequences $\{t^i_-\}_{i=1}^{\infty}$ and $\{t^i_+\}_{i=1}^{\infty}$ satisfying for $i = 1, 2, \dots,$

$$n_C(p, t^i_-) = \lfloor n^*_{SD_{k_p}} \rfloor, \qquad n_C(p, t^i_+) = \lceil n^*_{SD_{k_p}} \rceil.$$

We take this result as the base of our control design in the next section. We also note that the above convergence result holds "almost surely", that is it happens with probability one. Correspondingly, the results of this section hold also almost surely; however, we drop this term in our statements in the rest of the chapter for sake of simplicity.

Now given a population of the fixed size of n agents, each of which belongs to one of the types $SD_1, \dots, SD_l$ and PD, we study whether it is possible to make the number of cooperators equal to some reference value r for all time greater than some time τ. While there may be several ways to achieve this goal, e.g., fixing the strategies of some of the agents to cooperation, we take the type distribution of the agents as the control input. This is achieved by modifying the parameters of the agents' payoff matrices. By changing the types of a group of agents, a (new) type vector is acquired which leads to a possibly new number of cooperators in the final state. The interesting point is that since the long run behavior of the population state is independent of the initial strategies of the agents, it does not matter when and in which order the changes in the types are applied. Just when all changes take place, the number of cooperators reaches the new final state after some

time τ. So we ignore when the type modifications are applied, and shape our research question as follows:

Problem 1. (Reachable number of cooperators): *Consider the temper n^*, type vector ξ and initial population state $x(0)$. For a given nonnegative integer r and under the update rule (6.2), is it possible that by just changing the types of some of the agents, the number of cooperators equals r for all time greater than some constant τ?*

Each set of changes results in a (possibly new) type vector p. So to determine all of those r providing a positive answer to the above problem, we start with defining the *reachable set* $\mathcal{D}_n$ as all nonnegative integers r for which there exists a type vector p so that under the update rule (6.2), the number of cooperators equals and remains r after some time τ, i.e.,

$$\mathcal{D}_n := \big\{ r \in \mathbb{Z}_{\geq 0} \,|\, \exists p \in \mathcal{P}_n : (\exists \tau : n_C(p,t) = r \quad \forall t \geq \tau) \big\}.$$

D_n is completely determined by the following proposition.

Proposition 6.4. *It holds that*

$$\mathcal{D}_n = \left\{0, 1, \ldots, \min\left\{ \lfloor n^*_{SD_1} \rfloor, n \right\}\right\}. \tag{6.58}$$

Proof. First it is shown that if $r > n^*_{SD_1}$, then $r \notin \mathcal{D}_n$. Use contradiction. Assume that there exists a type vector $p \in \mathcal{P}_n$ such that

$$\exists \tau : \quad n_C(p,t) = r \quad \forall t \geq \tau. \tag{6.59}$$

According to Corollary 6.4, one of the followings holds

$$\exists \tau : \quad n_C(p,t) = \sum\nolimits_{i=1}^{k_p - 1} p_i \quad \forall t \geq \tau, \tag{6.60}$$

$$\exists \tau : \quad n_C(p,t) \in \left\{ \lfloor n^*_{SD_{k_p}} \rfloor, \lceil n^*_{SD_{k_p}} \rceil \right\} \quad \forall t \geq \tau. \tag{6.61}$$

If (6.60) is in force, we have that $r = \sum_{i=1}^{k_p - 1} p_i$. Hence,

$$\sum_{i=1}^{k_p - 1} p_i > n^*_{SD_1} \overset{(6.5)}{\Longrightarrow} \sum_{i=1}^{k_p - 1} p_i > n^*_{SD_{k_p - 1}}$$

which is in contradiction with the definition of k_p. On the other hand, if (6.61) holds, according to Corollary 6.4, n_C fluctuates between $\lfloor n^*_{SD_{k_p}} \rfloor$ and $\lceil n^*_{SD_{k_p}} \rceil$ in the long run. So in view of (6.59),

$$\lfloor n^*_{SD_{k_p}} \rfloor = \lceil n^*_{SD_{k_p}} \rceil = n^*_{SD_{k_p}} = r.$$

But this is in contradiction with the assumption $r > n^*_{SD_1}$. Hence,

$$r \notin \mathcal{D}_n \qquad \forall r > n^*_{SD_1}. \tag{6.62}$$

Now note that the number of cooperators cannot exceed the total number of agents in the population, i.e., n. Thus, $r \leq n$. This together with (6.62) confine $\mathcal{D}_n$ as follows

$$\mathcal{D}_n \subseteq \left\{0, 1, \ldots, \min\left\{ \lfloor n^*_{SD_1} \rfloor, n \right\}\right\}. \tag{6.63}$$

To complete the proof, we show that any r that belongs to $\left\{0,1,\ldots,\min\left\{\lfloor n^*_{SD_1}\rfloor, n\right\}\right\}$, also belongs to $\mathcal{D}_n$. Consider the case when $\lceil n^*_{SD_l}\rceil \leq n$. If

$$r \in \left\{\lceil n^*_{SD_l}\rceil, \lceil n^*_{SD_l}\rceil + 1, \ldots, \min\{\lfloor n^*_{SD_1}\rfloor, n\}\right\}, \tag{6.64}$$

then there exists an index $b \in \{1,\ldots,l\}$ such that $r = n^*_{SD_b}$ or there exists an index $b \in \{2,\ldots,l\}$ such that $n^*_{SD_{b-1}} > r > n^*_{SD_b}$. In any case, consider the following type vector

$$q = (\underbrace{0,\ldots,0}_{b-1}, r, 0, \ldots, 0, n-r).$$

It holds that

$$\left.\begin{array}{ll} \displaystyle\sum_{i=1}^{j} q_i < n^*_{SD_{b-1}} & \forall j \in \{1,\ldots,b-1\} \\ \displaystyle\sum_{i=1}^{b} q_i \geq n^*_{SD_b} & \end{array}\right\} \Rightarrow k_q = b.$$

Since $\sum_{i=1}^{b-1} q_i = 0 < n^*_{SD_b}$, in view of Corollary 6.4, there exists a time τ such that $n_C(q,t) = \sum_{i=1}^{b} q_i = r$ for all $t \geq \tau$. Hence, $r \in \mathcal{D}_n$. Now if

$$r \in \left\{0,1,\ldots,\lceil n^*_{SD_l}\rceil - 1\right\}\right\}, \tag{6.65}$$

then consider the following type vector

$$q' = (0,\ldots,0,r,n-r). \tag{6.66}$$

Clearly $\sum_{i=1}^{l} q'_i = r < n^*_{SD_l}$ and hence $k_{q'} = l+1$. So in view of Corollary 6.4, there exists a time τ such that $n_C(q',t) = \sum_{i=1}^{l} q'_i = r$ for all $t \geq \tau$. Therefore, $r \in \mathcal{D}_n$. Since (6.64) and (6.65) completely cover the set $\left\{0,1,\ldots,\min\left\{\lfloor n^*_{SD_1}\rfloor, n\right\}\right\}$, and for each case, it is shown that $r \in \mathcal{D}_n$, equation (6.58) is concluded in this case.

Now consider the case when $\lceil n^*_{SD_l}\rceil > n$. Then $\min\left\{\lfloor n^*_{SD_1}\rfloor, n\right\} = n$ and we need to show that if $r \in \{0,1,\ldots,n\}$, then $r \in \mathcal{D}_n$. This can be shown using q' in (6.66). So in general, it is proven that if $r \in \{0,1,\ldots,\min\left\{\lfloor n^*_{SD_1}\rfloor, n\right\}\}$ then $r \in \mathcal{D}_n$. Hence,

$$\left\{0,1,\ldots,\min\left\{\lfloor n^*_{SD_1}\rfloor, n\right\}\right\} \subseteq \mathcal{D}_n$$

which together with (6.63) complete the proof. □

Now that the reachable set $\mathcal{D}_n$ is clarified, we know to which variables we are potentially able to set the number of cooperators in the long run. However, it is still not clear how the types of the agents must change. As previously mentioned, each set of changes in the types of the agents results in a possibly new type vector. Therefore, if we know that (or those) feasible type vector(s) p under which the number of cooperators equals the reference value for all time greater than some constant τ, we also know the type of which agent should be changed to what, by just comparing the original type vector ξ with p. For a population of size n and a reference $r \in \mathcal{D}_n$, define $\mathcal{F}(r,n)$ as

the set of all feasible type-populations $p \in \mathcal{P}_n$ such that under the updater rule (6.2), the number of cooperators equals r for all time greater than some constant τ, i.e.,

$$\mathcal{F}(r,n) := \big\{p \,|\, p \in \mathcal{P}_n, (\exists \tau : n_C(p,t) = r \quad \forall t \geq \tau)\big\}.$$

Our goal is to determine $\mathcal{F}(r,n)$. Then it becomes clear how the types of the agents must change to have r cooperators in the long run. We need to define the following sets for a given $n \in \mathbb{N}$, $r \in \mathcal{D}_n$ and $b \in \{1,\dots,l+1\}$:

$$\mathcal{F}_1^b(r,n) := \big\{p \,|\, p \in \mathcal{P}_n, k_p = b, \sum_{i=1}^{b-1} p_i = r\big\},$$

$$\mathcal{F}_2^b(r,n) := \big\{p \,|\, p \in \mathcal{P}_n, k_p = b, \sum_{i=1}^{b-1} p_i < r\big\}.$$

Theorem 6.4. *Given the temper n^*, population size n and reference value $r \in \mathcal{D}_n$,*

1. *if there exists $b \in \{2,\dots,l\}$ such that $n^*_{SD_{b-1}} > r > n^*_{SD_b}$, then $\mathcal{F}(r,n) = \mathcal{F}_1^b(r,n)$;*
2. *if there exists $b \in \{1,\dots,l\}$ such that $r = n^*_{SD_b}$, then $\mathcal{F}(r,n) = \mathcal{F}_1^b(r,n) \cup \mathcal{F}_2^b(r,n)$;*
3. *if $r \in \big\{0,1,\dots,\min\big\{\lceil n^*_{SD_l}\rceil - 1, n\big\}\big\}$, then $\mathcal{F}(r,n) = \mathcal{F}_1^{l+1}(r,n)$.*

Proof. Let $p \in \mathcal{F}(r,n)$. Because of the definition of $\mathcal{F}(r,n)$ and the fact that r belongs to the reachable set $\mathcal{D}_n$, it holds that

$$\exists \tau : \quad n_C(p,t) = r \quad \forall t \geq \tau. \tag{6.67}$$

On the other hand, according to Corollary 6.4, one of the followings takes place

$$\exists \tau : \quad n_C(p,t) = \sum_{i=1}^{k_p-1} p_i \quad \forall t \geq \tau, \tag{6.68}$$

$$\exists \tau : \quad n_C(p,t) \in \big\{\lfloor n^*_{SD_{k_p}}\rfloor, \lceil n^*_{SD_{k_p}}\rceil\big\} \quad \forall t \geq \tau. \tag{6.69}$$

Now we proceed to the three cases of the theorem:

1. $\exists b \in \{2,\dots,l\} : \; n^*_{SD_{a-1}} > r > n^*_{SD_b}$.

First note that $k_p = l+1$ does not hold since then

$$r = \sum_{i=1}^{l} p_i > n^*_{SD_b},$$

which is in contradiction with the condition for k_p being equal to $l+1$, i.e., $\sum_{i=1}^{l} p_i < n^*_{SD_l}$. Now if (6.69) is in force, then in view of Corollary 6.4, n_C fluctuates between $\lfloor n^*_{SD_{k_p}}\rfloor$ and $\lceil n^*_{SD_{k_p}}\rceil$ in the long run. Hence, according to (6.67),

$$\lfloor n^*_{SD_{k_p}}\rfloor = \lceil n^*_{SD_{k_p}}\rceil = n^*_{SD_{k_p}} = r.$$

However, in view of (6.5), this is in contradiction with the assumption $n^*_{SD_{b-1}} > r > n^*_{SD_b}$. Thus, (6.68) holds. On the other hand, due to Corollary 6.4, (6.68) holds either if $k_p = l+1$, which is impossible, or if $\sum_{i=1}^{k_p-1} p_i \geq n^*_{SD_{k_p}}$. Then because of the definition of k_p, it holds that

$$\left.\begin{array}{c} n^*_{SD_{k_p-1}} > \sum_{i=1}^{k_p-1} p_i \geq n^*_{SD_{k_p}} \\ (6.67), (6.68): \sum_{i=1}^{k_p-1} p_i = r \end{array}\right\} \Rightarrow n^*_{SD_{k_p-1}} > r \geq n^*_{SD_{k_p}}.$$

By comparing this equation with $n^*_{SD_{b-1}} > r > n^*_{SD_b}$, we get $k_p = b$. Hence, $\sum_{i=1}^{k_p-1} p_i = \sum_{i=1}^{b-1} p_i = r$. So $p \in \mathcal{F}_1^a$, implying that $\mathcal{F}(r,n) \subseteq \mathcal{F}_1^b(r,n)$. On the other hand, for any $q \in \mathcal{F}_1^b(r,n)$, the following holds according to Corollary 6.4

$$\exists \tau: \quad n_C(q,t) = r \quad \forall t \geq \tau.$$

Thus, $q \in \mathcal{F}(r,n)$ and hence $\mathcal{F}_1^b(r,n) \subseteq \mathcal{F}(r,n)$. Therefore, we conclude that $\mathcal{F}(r,n) = \mathcal{F}_1^a$ which proves the first part of the theorem.

2. $\exists b \in \{1,\ldots,l\}: \; r = n^*_{SD_b}$.

First note that $k_p = l+1$ does not hold since then $r = \sum_{i=1}^{l} p_i = n^*_{SD_b}$ which is in contradiction with the condition for $k_p = l+1$, i.e., $\sum_{i=1}^{l} p_i < n^*_{SD_l}$. Hence, $k_p \neq l+1$. According to Corollary 6.4, $\mathcal{F}(r,n) \subseteq \mathcal{F}_1^b(r,n) \cup \mathcal{F}_2^b(r,n)$. On the other hand, similar to the previous part, if $q \in \mathcal{F}_1^b(r,n)$, then $q \in \mathcal{F}(r,n)$ and hence $\mathcal{F}_1^b(r,n) \subseteq \mathcal{F}(r,n)$. So we only need to prove that $\mathcal{F}_2^b(r,n) \subseteq \mathcal{F}(r,n)$. Consider $q \in \mathcal{F}_2^b(r,n)$. Since $k_q = b$ and $\sum_{i=1}^{b-1} q_i < r$, in view of Corollary 6.4,

$$\exists \tau: \quad n_C(q,t) \in \left\{ \lfloor n^*_{SD_{k_q}} \rfloor, \lceil n^*_{SD_{k_q}} \rceil \right\} \quad \forall t \geq \tau. \tag{6.70}$$

Since $n^*_{SD_b} = r \in \mathbb{Z}$ and $k_q = b$, it holds that $\lfloor n^*_{SD_{k_q}} \rfloor = \lceil n^*_{SD_{k_q}} \rceil = n^*_{SD_{k_q}} = r$. Hence, (6.70) implies

$$\exists \tau: \quad n_C(q,t) = r \quad \forall t \geq \tau.$$

Hence, $q \in \mathcal{F}(r,n)$, implying $\mathcal{F}_2^b(r,n) \subseteq \mathcal{F}(r,n)$, which completes the proof of the second part of the theorem.

3. $r \in \left\{0, 1, \ldots, \min\left\{\lceil n^*_{SD_l} \rceil - 1, n\right\}\right\}$.

Clearly (6.69) does not hold in this case; otherwise,

$$n^*_{SD_{k_p}} = r \leq \min\left\{\lceil n^*_{SD_l} \rceil - 1, n\right\} \leq \lceil n^*_{SD_l} \rceil - 1 < n^*_{SD_l} \tag{6.71}$$

which is in contradiction with (6.5). Therefore, (6.68) is in force. Then in view of (6.67), it holds that $\sum_{i=1}^{k_p-1} p_i = r$. On the other hand, according to Corollary 6.4, (6.68) holds only if

$$\sum_{i=1}^{k_p-1} p_i \geq n^*_{SD_{k_p}} \quad \text{or} \quad k_p = l+1.$$

The first yields a contradiction since in view of (6.71),

$$\sum_{i=1}^{k_p-1} p_i = r < n^*_{SD_l} \xRightarrow{(6.5)} \sum_{i=1}^{k_p-1} p_i < n^*_{SD_{k_p}}.$$

The second case, i.e., $k_p = l+1$, results in $\sum_{i=1}^{l} p_i = r$. Hence, $p \in \mathcal{F}_1^{l+1}(r,n)$. So $\mathcal{F}(r,n) \subseteq \mathcal{F}_1^{l+1}(r,n)$. Now if $q \in \mathcal{F}_1^{l+1}(r,n)$, then $k_p = l+1$ and in view of Corollary 6.4, since $\sum_{i=1}^{l} q_i = r$, the following holds

$$\exists \tau : \quad n_C(q,t) = r \quad \forall t \geq \tau.$$

Hence, $q \in \mathcal{F}(r,n)$, implying that $\mathcal{F}(r,n) = \mathcal{F}_1^{l+1}(r,n)$, which completes the proof of the third part of the theorem and hence the whole. □

Now that $\mathcal{F}(r,n)$ is determined, it becomes clear how to change the types of the agents to set the final state of the number of cooperators to r. However, as $\mathcal{F}(r,n)$ is a set, there may be several ways to change the types of the agents and some may require more changes than the others. Note that by *a change*, we mean the type transition of some agent i from her current type to another. Some of the changes in the types do not affect the number of cooperators in the long run at all. For example, in case the number of cooperators has reached an equilibrium point $n^*_{SD_{k_p}}$, by changing the type of an SD_{k_p+1} agent to SD_{k_p+2}, still n_C reaches and stays at the equilibrium $n^*_{SD_{k_p}}$ after some finite time τ. We are not interested in these kind of changes. In general, we seek the minimum number of changes needed to set the number of cooperators to r at the final state.

Problem 2. (Minimum type-change cooperation control) *Given the temper n^*, type vector $\xi \in \mathcal{P}_n$ and reference value $r \in \mathcal{D}_n$, what is the minimum number of agents needed to change their types to ensure that under the update rule* (6.2)*, the number of cooperators equals r for all time greater than some constant τ? Moreover, the types of which agents should be changed and to what types they should be changed?*

We can reformulate Problem 2 as the following optimization problem.

Problem 3. *Given the temper n^*, type vector $\xi \in \mathcal{P}_n$ and reference value $r \in \mathcal{D}_n$,*

$$m := \arg\min_{p \in \mathcal{P}_n} \|p - \xi\|_1 \quad \text{s.t.} \quad \exists \tau : \quad n_C(p,t) = r \quad \forall t \geq \tau$$

where $\|\cdot\|_1$ is the 1-norm, i.e., $\|x\|_1 = \sum_i |x_i|$.

Now m provides the minimum number of changes in Problem 2. Moreover, each optimal solution p is a type vector that (i) guaranties the number of cooperators becomes r for all time greater than some τ and (ii) among all other type vectors with such a property, p has the minimum difference from the given type vector ξ. Therefore, by comparing p with ξ, it becomes clear how the types of the agents must change. For example, if $\xi = (5,4)$ and $p = (8,1)$, we know that the types of 3 PD agents must be changed to SD_1 in order to make the number of cooperators equal to the desired value r in the long run. For a reachable r, the following theorem determines m and the way the types must change.

Theorem 6.5. *Consider the temper n^*, type vector $\xi \in \mathcal{P}_n$ and reference value $r \in \mathcal{D}_n$. Then*

1. *if there exists* $b \in \{2, \dots, l\}$ *such that* $n^*_{SD_{b-1}} > r > n^*_{SD_b}$, *then* $m = |r - \sum_{i=1}^{b-1} \xi_i|$. *Moreover, when* $r - \sum_{i=1}^{b-1} \xi_i > 0$, *the minimum is achieved if and only if* $r - \sum_{i=1}^{b-1} \xi_i$ *agents are chosen from the types* $SD_b, \dots, SD_l$ *and* PD, *and the type of each of these agents is changed to one of the types* $SD_1, \dots, SD_{b-1}$. *When* $r - \sum_{i=1}^{b-1} \xi_i < 0$, *the minimum is achieved if and only if* $r - \sum_{i=1}^{b-1} \xi_i$ *agents are chosen from the types* $SD_1, \dots, SD_{b-1}$, *and the type of each of these agents is changed to one of the types* $SD_b, \dots, SD_l$ *or* PD.

2. *If there exists* $b \in \{1, \dots, l\}$ *such that* $n^*_{SD_b} = r$, *then*

   (a) *if* $\sum_{i=1}^{b} \xi_i \geq n^*_{SD_{b-1}}$ *and* $\sum_{i=1}^{b-1} \xi_i \leq n^*_{SD_b}$, *then* $m = \sum_{i=1}^{b} \xi_i - \lceil n^*_{SD_{b-1}} \rceil + 1$ *and the minimum is achieved if and only if* $\sum_{i=1}^{b} \xi_i - \lceil n^*_{SD_{b-1}} \rceil + 1$ *agents are chosen from the types* $SD_1, \dots, SD_b$, *and the type of each of these agents is changed to one of the types* $SD_{b+1}, \dots, SD_l$ *or* PD;

   (b) *if* $\sum_{i=1}^{b} \xi_i \geq n^*_{SD_{b-1}}$ *and* $\sum_{i=1}^{b-1} \xi_i > n^*_{SD_b}$, *then if* $\xi_b + n^*_{SD_b} < n^*_{SD_{b-1}}$, *it holds that* $m = \sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}$, *and the minimum is achieved if and only if* $\sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}$ *agents are chosen from the types* $SD_1, \dots, SD_{b-1}$, *and the type of each of these agents is changed to one of the types* $SD_b, \dots, SD_l$ *or* PD; *on the other hand, if* $\xi_b + n^*_{SD_b} \geq n^*_{SD_{b-1}}$, *it holds that* $m = \sum_{i=1}^{b} \xi_i - \lceil n^*_{SD_{b-1}} \rceil + 1$, *and the minimum is achieved if and only if in addition to the above changes,* $\xi_b + n^*_{SD_b} - \lceil n^*_{SD_{b-1}} \rceil + 1$ *agents are chosen from the types* $SD_1, \dots, SD_b$, *and the type of each of these agents is changed to one of the types* $SD_{b+1}, \dots, SD_l$ *or* PD;

   (c) *if* $\sum_{i=1}^{b} \xi_i < n^*_{SD_b}$, *then* $m = n^*_{SD_b} - \sum_{i=1}^{b} \xi_i$, *and the minimum is achieved if and only if* $n^*_{SD_b} - \sum_{i=1}^{b} \xi_i$ *agents are chosen from the types* $SD_{b+1}, \dots, SD_l$ *and* PD, *and the type of each of these agents is changed to one of the types* $SD_1, \dots, SD_b$;

   (d) *if* $n^*_{SD_{b-1}} > \sum_{i=1}^{b} \xi_i \geq n^*_{SD_b}$ *and* $\sum_{i=1}^{b-1} \xi_i \leq n^*_{SD_b}$, *then* $m = 0$, *i.e., no changes are required;*

   (e) *if* $n^*_{SD_{b-1}} > \sum_{i=1}^{b} \xi_i \geq n^*_{SD_b}$ *and* $\sum_{i=1}^{b-1} \xi_i > n^*_{SD_b}$, *then* $m = \sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}$, *and the minimum is achieved if and only if* $\sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}$ *agents are chosen from the types* $SD_1, \dots, SD_{b-1}$, *and the type of each of these agents is changed to one of the types* $SD_b, \dots, SD_l$ *and* PD.

3. *If* $r \in \left\{0, 1, \dots, \min\left\{\lceil n^*_{SD_l} \rceil - 1, n\right\}\right\}$, *then* $m = |r - \sum_{i=1}^{l} \xi_i|$. *Moreover, when* $r - \sum_{i=1}^{l} \xi_i > 0$, *the minimum is achieved if and only if the types of* $r - \sum_{i=1}^{l} \xi_i$ *number of the* PD *agents are changed to* SD. *When* $r - \sum_{i=1}^{l} \xi_i < 0$, *the minimum is achieved if and only if the types of* $r - \sum_{i=1}^{l} \xi_i$ *number of the* SD *agents are changed to* PD.

Proof. Let p denote the type vector after applying the minimum number of changes to the types of the agents in order to have r cooperators for all time greater than some constant τ. Then $p \in \mathcal{F}(r, n)$. We proceed to the three cases of the theorem:

1. $\exists b \in \{2, \dots, l\} : \; n^*_{SD_{b-1}} > r > n^*_{SD_b}$.

According to Theorem 6.4, $p \in \mathcal{F}_1^b(r,n)$ in this case. By comparing ξ with any $p \in \mathcal{F}_1^b(r,n)$, it becomes clear that at least $|r - \sum_{i=1}^{b-1} \xi_i|$ changes are required:

$$\|p - \xi\|_1 = \sum_{i=1}^{l+1} |p_i - \xi_i| \geq \sum_{i=1}^{b-1} |p_i - \xi_i| \geq |r - \sum_{i=1}^{b-1} \xi_i|.$$

Hence,

$$m \geq |r - \sum_{i=1}^{b-1} \xi_i|. \tag{6.72}$$

Now if $r - \sum_{i=1}^{b-1} \xi_i > 0$, the number of the agents belonging to the types $SD_1, \ldots, SD_{b-1}$ is less than the desired number. So the type of $r - \sum_{i=1}^{b-1} \xi_i$ of the agents belonging to any other type, i.e., $SD_b, \ldots, SD_l$ or PD have to be changed to any of the types $SD_1, \ldots, SD_{b-1}$. This proves the "only if" part. Now let q denote the type vector after $r - \sum_{i=1}^{b-1} \xi_i$ agents of the types $SD_b, \ldots, SD_l$ or PD are chosen and the types of each of them are changed to one of $SD_1, \ldots, SD_{b-1}$. Then $\sum_{i=1}^{b-1} q_i = r$. Thus, because of $n^*_{SD_{b-1}} > r > n^*_{SD_b}$, it holds that

$$\left.\begin{array}{ll} \sum_{i=1}^{j} q_i \leq r < n^*_{SD_{b-1}} & \forall j \in \{1, \ldots, b-1\} \\ \sum_{i=1}^{b} q_i \geq r > n^*_{SD_b} & \end{array}\right\} \Rightarrow k_q = b.$$

So in view of Corollary 6.4, there exists a time τ such that $n_C(q,t) = \sum_{i=1}^{b-1} q_i = r$ for all $t \geq \tau$. This proves the "if" part, and also implies that $m \leq r - \sum_{i=1}^{b-1} \xi_i$. A similar reasoning holds for the case when $r - \sum_{i=1}^{b-1} \xi_i < 0$, implying that $m \leq \sum_{i=1}^{b-1} -r\xi_i$. Therefore, in any case, $m \leq |r - \sum_{i=1}^{b-1} \xi_i|$. Hence, in view of (6.72), $m = |r - \sum_{i=1}^{b-1} \xi_i|$, and the proof of this case is complete.

2. $\exists b \in \{1, \ldots, l\} : \; n^*_{SD_b} = r$.

According to Theorem 6.4, $p \in \mathcal{F}(r,n) = \mathcal{F}_1^b(r,n) \cup \mathcal{F}_2^b(r,n)$. Then

$$k_p = b \Rightarrow \sum_{i=1}^{b} p_i < n^*_{SD_{b-1}} \Rightarrow \sum_{i=1}^{b} p_i \leq \lceil n^*_{SD_{b-1}} \rceil - 1. \tag{6.73}$$

Now we proceed to the following sub-cases:

(a) $\sum_{i=1}^{b} \xi_i \geq n^*_{SD_{b-1}}$ and $\sum_{i=1}^{b-1} \xi_i \leq n^*_{SD_b}$.

By comparing the inequality in (6.73) with $\sum_{i=1}^{b} \xi_i \geq n^*_{SD_{b-1}}$, it can be concluded that at least $\sum_{i=1}^{b} \xi_i - \lceil n^*_{SD_{b-1}} \rceil + 1$ agents from the types $SD_1, \ldots, SD_b$ must be chosen, and the types of each of them must be changed to one of the types $SD_{b+1}, \ldots, SD_l$ or PD. Hence, $m \geq \sum_{i=1}^{b} \xi_i - \lceil n^*_{SD_{b-1}} \rceil +$

1. Now let q denote the type vector after applying the above changes. Then $\sum_{i=1}^{b} q_i = \lceil n^*_{SD_{b-1}} \rceil - 1$. So

$$\left.\begin{aligned} &\sum_{i=1}^{j} q_i < n^*_{SD_{b-1}} \quad \forall j \in \{1, \ldots, b-1\} \\ (6.5): &\sum_{i=1}^{b} q_i \geq n^*_{SD_b} \end{aligned}\right\} \Rightarrow k_q = b.$$

On the other hand, since the number of the SD_i agents, $i = 1, \ldots, b-1$, has not increased after applying the changes, it holds that

$$\sum_{i=1}^{b-1} q_i \leq \sum_{i=1}^{b-1} \xi_i \leq n^*_{SD_b}.$$

Hence, in view of Corollary 6.4, there exists a time τ such that $n_C(q,t) = n^*_{SD_b}$ for all $t \geq \tau$. Therefore, $m = \sum_{i=1}^{b} \xi_i - \lceil n^*_{SD_{b-1}} \rceil + 1$ and the proof of this case is complete.

(b) $\sum_{i=1}^{b} \xi_i \geq n^*_{SD_{b-1}}$ and $\sum_{i=1}^{b-1} \xi_i > n^*_{SD_b}$.

According to Theorem 6.4, $p \in \mathcal{F}(r,n) = \mathcal{F}_1^b(r,n) \cup \mathcal{F}_2^b(r,n)$. Hence, $\sum_{i=1}^{b-1} p_i \leq n^*_{SD_b}$. By comparing this with $\sum_{i=1}^{b-1} \xi_i > n^*_{SD_b}$, it can be concluded that at least $\sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}$ agents from the types $SD_1, \ldots, SD_{b-1}$ must be chosen and the types of each of them must be changed to one of the types $SD_b, \ldots, SD_l$ or PD. Let d denote the type vector after making these changes. Then $\sum_{i=1}^{b-1} d_i = n^*_{SD_b} < n^*_{SD_{b-1}}$. On the other hand,

$$\sum_{i=1}^{b} d_i = \sum_{i=1}^{b} \xi_i - (\sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}) = \xi_b + n^*_{SD_b}. \tag{6.74}$$

So $\sum_{i=1}^{b} d_i \geq n^*_{SD_b}$. Now if $\xi_b + n^*_{SD_b} < n^*_{SD_{b-1}}$, then by (6.74), $\sum_{i=1}^{b} d_i < n^*_{SD_{b-1}}$, and hence $\sum_{i=1}^{b-1} d_i < n^*_{SD_{b-1}}$. Hence, $k_d = b$. Moreover, because of $\sum_{i=1}^{b-1} d_i = n^*_{SD_b}$, it holds that $d \in \mathcal{F}_1^b(r,n) \subseteq \mathcal{F}(r,n)$ and hence no more changes are required. Thus, $m = \sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}$ for this case.

If $\xi_b + n^*_{SD_b} \geq n^*_{SD_{b-1}}$, then by (6.74), $\sum_{i=1}^{b} d_i \geq n^*_{SD_{b-1}}$. Hence, according to (6.73), the desired type vector p has the property that $\sum_{i=1}^{b} p_i < \lceil n^*_{SD_{b-1}} \rceil - 1$. So still at least $\sum_{i=1}^{b} d_i - \lceil n^*_{SD_{b-1}} \rceil + 1$ agents from the types $SD_1, \ldots, SD_b$ must be chosen and the type of each of them must be changed to one of types $SD_{b+1}, \ldots, SD_l$ or PD. Therefore,

$$\begin{aligned} m &\geq \left(\sum_{i=1}^{b-1} \xi_i - n^*_{SD_b}\right) + \left(\sum_{i=1}^{b} d_i - \lceil n^*_{SD_{b-1}} \rceil + 1\right) \\ &\overset{(6.74)}{=} \sum_{i=1}^{b} \xi_i - \lceil n^*_{SD_{b-1}} \rceil + 1. \end{aligned}$$

Now let q denote the type vector after changing the types of $\sum_{i=1}^{b-1}\xi_i - n^*_{SD_b}$ number of $SD_1,\dots,SD_{b-1}$ agents to $SD_b,\dots,SD_l$ and PD and changing the types of $\xi_b + n^*_{SD_b} - \lceil n^*_{SD_{b-1}}\rceil + 1$ number of $SD_1,\dots,SD_b$ agents to $SD_{b+1},\dots,SD_l$ and PD. Then $\sum_{i=1}^{b} q_i = n^*_{SD_b}$ and $\sum_{i=1}^{b-1} q_i \leq n^*_{SD_b}$. Hence, $k_q = b$ and in view of Corollary 6.4, there exists a time τ such that $n_C(q,t) = n^*_{SD_b}$ for all $t \geq \tau$. Therefore, $m = \sum_{i=1}^{b}\xi_i - \lceil n^*_{SD_{b-1}}\rceil + 1$ and the proof of this case is complete.

(c) $\sum_{i=1}^{b}\xi_i < n^*_{SD_b}$.

According to Theorem 6.4, $p \in \mathcal{F}(r,n) = \mathcal{F}_1^b(r,n)\cup\mathcal{F}_2^b(r,n)$. Hence, $k_p = b$, resulting in $\sum_{i=1}^{b} p_i \geq n^*_{SD_b}$. By comparing this with $\sum_{i=1}^{b}\xi_i < n^*_{SD_b}$, it can be concluded that at least $n^*_{SD_b} - \sum_{i=1}^{b}\xi_i$ agents from the types $SD_{b+1},\dots,SD_l$ and PD must be chosen and the types of each of them must be changed to one of the types $SD_1,\dots,SD_b$. So $m \geq n^*_{SD_b} - \sum_{i=1}^{b}\xi_i$. Now let q denote the type vector after making the above changes. Then $\sum_{i=1}^{b} q_i = n^*_{SD_b}$. Moreover,

$$\sum_{i=1}^{b-1} q_i \leq \sum_{i=1}^{b} q_i = n^*_{SD_b}. \tag{6.75}$$

Hence, in view of (6.5), $\sum_{i=1}^{b-1} q_i < n^*_{SD_{b-1}}$, implying that $k_q = b$. So in view of Corollary 6.4 and (6.75), there exists a time τ such that $n_C(q,t) = n^*_{SD_b}$ for all $t \geq \tau$. Therefore, $m = n^*_{SD_b} - \sum_{i=1}^{b}\xi_i$, and the proof of this case is complete.

(d) $n^*_{SD_{b-1}} > \sum_{i=1}^{b}\xi_i \geq n^*_{SD_b}$ and $\sum_{i=1}^{b-1}\xi_i \leq n^*_{SD_b}$.

Since $n^*_{SD_{b-1}} > \sum_{i=1}^{b}\xi_i \geq n^*_{SD_b}$ it holds that $k_\xi = b$. Now either if $\sum_{i=1}^{b-1}\xi_i = n^*_{SD_b}$ or if $\sum_{i=1}^{b-1}\xi_i < n^*_{SD_b}$, in view of Corollary 6.4, there exists a time τ such that $n_C(\xi,t) = n^*_{SD_b}$ for all $t \geq \tau$. Therefore, no changes are needed and hence $m = 0$.

(e) $n^*_{SD_{b-1}} > \sum_{i=1}^{b}\xi_i \geq n^*_{SD_b}$ and $\sum_{i=1}^{b-1}\xi_i > n^*_{SD_b}$.

According to Theorem 6.4, $p \in \mathcal{F}(r,n) = \mathcal{F}_1^b(r,n)\cup\mathcal{F}_2^b(r,n)$. Hence, $\sum_{i=1}^{b-1} p_i \leq n^*_{SD_b}$. Since $\sum_{i=1}^{b-1}\xi_i > n^*_{SD_b}$, it can be concluded that at least $\sum_{i=1}^{b-1}\xi_i - n^*_{SD_b}$ agents from the types $SD_1,\dots,SD_{b-1}$ must be chosen and the types of each of them must be changed to one of the types $SD_b,\dots,SD_l$ and PD. So $m \geq \sum_{i=1}^{b-1}\xi_i - n^*_{SD_b}$. Now let q denote the type vector after making the above changes. Then $\sum_{i=1}^{b-1} q_i = n^*_{SD_b}$. Hence,

$$\left.\begin{aligned} &\sum_{i=1}^{j} q_i \leq n^*_{SD_b} \overset{(6.5)}{<} n^*_{SD_{b-1}} \quad \forall j \in \{1,\dots,b-1\} \\ &\sum_{i=1}^{b} q_i \geq \sum_{i=1}^{b-1} q_i = n^*_{SD_b} \end{aligned}\right\} \Rightarrow k_q = b.$$

Now because of $\sum_{i=1}^{b-1} q_i = n^*_{SD_b}$, in view of Corollary 6.4, there exists a time τ such that $n_C(q,t) = \sum_{i=1}^{b-1} q_i = n^*_{SD_b}$ for all $t \geq \tau$. Therefore, $m = \sum_{i=1}^{b-1}\xi_i - n^*_{SD_b}$, and the proof of this case is complete.

3. $r \in \{0, 1, \ldots, \min\{\lceil n^*_{SD_l} \rceil - 1, n\}\}$.

According to Theorem 6.4, $p \in \mathcal{F}(r,n) = \mathcal{F}_1^{l+1}(r,n)$. Hence, $\sum_{i=1}^{l} p_i = r$. By comparing this with $\sum_{i=1}^{l} \xi_i$, it can be concluded that $m \geq |r - \sum_{i=1}^{l} \xi_i|$. If $r - \sum_{i=1}^{l} \xi_i > 0$, then the types of $r - \sum_{i=1}^{l} \xi_i$ number of the PD agents must be changed to SD; otherwise, $r - \sum_{i=1}^{l} \xi_i$ number of the SD agents must be changed to PD. Now let q denote the type vector after the above $|r - \sum_{i=1}^{l} \xi_i|$ changes are made. Then

$$\sum_{i=1}^{l} q_i = r < \min\{\lceil n^*_{SD_l} \rceil - 1, n\} \leq \lceil n^*_{SD_l} \rceil - 1 \Rightarrow k_q = l + 1.$$

Hence, in view of Corollary 6.4, there exists a time τ such that $n_C(q,t) = \sum_{i=1}^{l} q_i = r$ for all $t \geq \tau$. Therefore, $m = |r - \sum_{i=1}^{l} \xi_i|$, and the proof of this case, and hence the whole is complete. □

According to the theorem, the number of changes in the types of the individuals to get a desired cooperation level in the population, does not necessarily depend on the current cooperation level of the population.

6.6 Concluding remarks

We have studied a finite heterogeneous population of decision-making agents under the myopic best-response update rule. We have shown that based on the type of the population, the total number of cooperators in the long run either becomes fixed or fluctuates between two consecutive numbers. We have used the convergence results to mathematically explain the level-off phenomenon, and have shown the existence of sustainable levels of cooperation where increasing the ratio of the less- or more-cooperative agents does not necessarily change the number of cooperators in the long run. This sheds light on how certain cooperation levels can be maintained in a population of selfish individuals. It can also explain why sometimes increasing the tendencies of some individuals in social groups does not lead to a higher level of cooperation. Moreover, the result indicates that if one wants to control the cooperation level of such populations by influencing the types of the agents, different choices of manipulated agents might lead to different outcomes. We have formulated such a control problem more formally in Section 6.5 and investigated how to control the number of cooperators, and in particular, for a given population and its types, we have shown how to change the types of the agents to reach a desired number of cooperators in the long run. Since many real networks are structured and not well-mixed, it is of great interest to extend the results to the case of structured populations, which is the topic of Chapter 9.

Chapter 7

Convergence of linear threshold decision-making dynamics in finite heterogeneous populations

Linear threshold models have been studied extensively in networked populations; however, less attention has been paid to the perception differences among the individuals of the population. To focus on this effect, we consider a finite well-mixed heterogeneous population where each individual is associated with a threshold in the form of a fixed ratio within zero and one that can be unique to this agent. Each agent is initialized with a random choice of the strategy A or B. Then according to their thresholds, the agents update their strategies asynchronously. More specifically, at each time step an agent is randomly activated to update; if the ratio of agents playing A is greater (resp. less) than her threshold, she updates to A (resp. B); if the ratio is equal to her threshold, she keeps her current strategy. Despite the simplicity of such a decision-making model, the population dynamics may exhibit several possible long-run behavior for the same initial condition. Indeed, we show that for any given initial condition, almost surely after a finite number of time steps, the population reaches an equilibrium where no agent's threshold is violated; however, the equilibrium is not necessarily uniquely determined by the initial condition, but depends on the activation sequence of the agents. We find all those possible equilibria that the dynamics may reach from a given initial condition, and show that in contrast to the case of homogeneous populations, the heterogeneity in agents' thresholds give rise to several equilibria where both A-playing and B-playing agents coexist. This confirms the possibility of having both types of players in the long-run, even in a well-mixed population. Such results highlight the crucial role of perception differences, characterized by thresholds, among the individuals or in general the heterogeneity of the population in linear threshold models for the coexistence of individuals playing competing strategies. We also find necessary and sufficient conditions for the asymptotic stability of the equilibrium states and show how to calculate the contagion probability for a population with two thresholds.

7.1 Introduction

After studying finite well-mixed populations of individuals playing snowdrift and prisoner's dilemma games and updating according to the best-response update rule, here we investigate the case when the individuals play coordination games, leading to linear threshold models. The spread of social innovations, technological innovations, viral infections and reforms of corporate governance, are examples of a cascading behavior where the adaption of an action by a portion of individuals makes it more likely for the action to spread to the rest of the population [119]. At the individual level, such a behavior can be modeled as follows: I adopt action A only if enough of my pears have done so. This is actually the essence of the so called *linear threshold models* originally framed by Mark Granovetter [53] where each individual has a *threshold* determining whether to adopt a specific action based on the current number of other individuals who have already adopted that action. On the other hand, in a different framework, *evolutionary game theory* [168, 94, 10, 118, 183, 83] postulates an equivalent mechanism to describe such cascading behavior, namely the *(myopic) best-response update rule* [149], resulting in the *best-response dynamics* [39, 45, 14, 155]. In this game theoretical representation, individuals interact with each other and earn payoffs determined by their strategies (actions) and utility functions that can be unique to each individual. Then each individual updates her strategy to the one that maximizes her payoff against the average population. In particular, when the payoffs satisfy those of a *coordination game*, the best-response dynamics becomes exactly equivalent to the linear threshold model. Indeed some recent literature confirms that human does follow the best-response update rule in coordination games [100].

Both the linear threshold model and the best-response update rule have been studied under different circumstances such as when the population is structured [187, 109, 110, 88, 92, 164, 61], the update is noisy [84, 9, 8], and other situations [148, 123, 7, 99]. Although these works reveal interesting aspects of the linear threshold models, e.g., how the topology of the interaction network affects the spread of a specific action [109], almost all of them consider a homogeneous population, that is when the thresholds of all individuals are the same. However, some studies have highlighted the impact of heterogeneity in the best-response dynamics by showing how small degrees of heterogeneity in the population leads to complex features such as cooperation sustainability (the level-off phenomenon) [139, 136]. Also in the experimental work [100], deviations from the best-response update rule are more likely due to individual heterogeneity rather than time or space. This motivates us to focus on the role of heterogeneity in the best-response update rule, or equivalently the linear threshold models. In an interesting work [2], linear threshold models in finite networks have been considered in structured heterogeneous populations, showing the existence of cycles in the behavior of the individuals when they update synchronously. Although the synchronous updating applies to situations where agents revise their choices simultaneously, the asynchronous update rule is perhaps more realistic, specially in scenarios where the consequences of the decisions occur faster than the decisions themselves.

The aim of this work is to investigate the effect of heterogeneity using linear threshold models. So to eliminate other effects such as population structures, we consider a finite well-mixed population of decision-making individuals having different thresholds. The individuals choose between two options A and B, and update asynchronously based on the linear threshold model. More specifically, at each time step, an individual is randomly activated to revise her strategy based on her threshold and the ratio of A-playing individuals in the population, resulting in population

dynamics. We take the distribution of the A-playing individuals among the agents with different thresholds as the state of the system. Similar to the anti-coordination best-response dynamics in Chapter 6, here we show that regardless of the initial state and the activation sequence, almost surely the dynamics reach an equilibrium state where one individual becomes a benchmark such that all others with higher tendencies to play A, play A, and all with lower tendencies play B. Several results have not been reported in the literature before and are the main contributions of this chapter. First, in sharp contrast to the anti-coordination or synchronous setup, the population dynamics may exhibit several possible long-run behavior for the same initial condition, that is for the same initial state, the final state may vary depending on the activation sequence of the agents. Moreover, in contrast to homogeneous setups, the dynamics may allow several equilibria where both A-playing and B-playing agents coexist. Therefore, it is possible to have both types of players in the long-run, even in a well-mixed population. These results highlight the key role of (perception) heterogeneity of the individuals in linear threshold models. In addition, we investigate the stability of the equilibria and show that clean-cut equilibrium states where individuals with a particular threshold either all play A or all play B, can be asymptotically stable under some conditions on the distribution of the thresholds, yet the other type of equilibrium states, which we refer to as ruffled, are unstable. Finally, by using a recursive equation we show how to calculate the contagion probability, that is the probability of reaching the state where every individual plays A, in a population with two different types of individuals.

The rest of the chapter is organized as follows. In Section II, we present the framework and formulate the problem. In Section III, we find the equilibrium states of the population dynamics. In Section IV, we provide the main convergence results of this chapter, which is followed by the stability analysis in Section V. In Section 7.6, we proceed to the contagion probability and finish the chapter in Section 7.7 with the concluding remarks.

7.2 Model

We provide two equivalent models for the problem studied in this chapter and start with the linear threshold model used in the rest of the chapter.

7.2.1 Linear threshold model

Consider a well-mixed population of n agents who choose one of the options A or B over a time sequence $t = 0, 1, \ldots$. Each agent $i \in \{1, \ldots, n\}$ has a time-invariant threshold $\tau_i \in (0, 1)$ and her choice (decision) at time t is denoted by $d_i(t)$. At every time step t, a random agent i is activated to revise her choice at $t + 1$. The linear threshold model dictates that agent i chooses A, if and only if a portion greater than her threshold has already chosen A. The choice of B is made if and only if in the population, a portion less than her threshold chooses A. In case when the portion exactly equals the threshold of agent i, she continues with her previous choice, allowing choice robustness for the agents. Namely, the update rule governing agent i's choice can be written as

$$d_i(t+1) = \begin{cases} A & x^A(t) > \tau_i \\ d_i(t) & x^A(t) = \tau_i \\ B & x^A(t) < \tau_i \end{cases} \tag{7.1}$$

where $x^A(t)$ denotes the ratio of agents in the population choosing A at t.

Different from the existing study on homogeneous populations where the evolution of x^A is the subject of interest, in this chapter for heterogeneous populations, we consider the challenging problem of how the distribution of the choices over agents with different thresholds changes over time. This can be captured by classifying the heterogeneous agents into different types according to their thresholds. Each agent's threshold lies in one of the intervals $(0, \frac{1}{n}], (\frac{1}{n}, \frac{2}{n}], \ldots, (\frac{n-1}{n}, 1]$. Denote those intervals that cover at least one agent's threshold by $(\frac{n_1^*-1}{n}, \frac{n_1^*}{n}], \ldots, (\frac{n_l^*-1}{n}, \frac{n_l^*}{n}]$ where

$$n_j^* < n_{j+1}^* \qquad j = 1, \ldots, l-1. \tag{7.2}$$

The equality case in (7.1) never takes place for an agent whose threshold lies in an open interval $(\frac{n_j^*-1}{n}, \frac{n_j^*}{n}), j \in \{1, \ldots, l\}$, whom we call a *type j agent*, but may take place for an agent whose threshold equals one of the ratios $\frac{n_j^*}{n}, j \in \{1, \ldots, l\}$, whom we call a *type j^* agent*. This leads to in total $2l$ different possible types of agents. Let n_j and $\tilde{n}_j$ denote the number of type j and type j^* agents. The heterogeneity of the population is then characterized by the $2l$-dimensional vector $(n_1, \tilde{n}_1, \ldots, n_l, \tilde{n}_l)$. Let $n_j^A(t)$ and $\tilde{n}_j^A(t)$ denote the number of type j and type j^* agents whose choices are A at time t. We stack all n_j^A and $\tilde{n}_j^A$ together to get the *population state*

$$x(t) \triangleq (n_1^A(t), \tilde{n}_1^A(t), \ldots, n_l^A(t), \tilde{n}_l^A(t))$$

which lies in the state space

$$\mathcal{X} \triangleq \{(a_1, b_1, \ldots, a_l, b_l) \,|\, a_i, b_i \in \mathbb{Z}_{\geq 0}, a_i \leq n_i, b_i \leq \tilde{n}_i, i = 1, \ldots, l\}.$$

The activation mechanism together with update rule (7.1) governs the dynamics of $x(t)$, which we refer to as the *population dynamics*. The main goal of this chapter is to determine the asymptotic behavior of $x(t)$ from any given initial condition $x(0)$. We make the following mild and standard assumption on the way the agents are activated. We assume that every activation sequence is *persistent*, that is for every agent i and every time t, with probability one there exists some finite time $t_i \geq t$ at which agent i is activated. Before proceeding to the analysis, we discuss an equivalent game theoretical representation of the above threshold model, which is similar to the game theoretical model presented in Chapter 6.

7.2.2 Game theoretical representation

Consider a well-mixed population of n agents that make decisions over a time sequence $t = 0, 1, \ldots$. Initially each agent starts with one of the two strategies A or B. As time evolves, the agents update their strategies asynchronously according to the payoffs they earn against the average population and under the best response update rule as explained in the following. Denote the strategy of agent i by $s_i(t)$ which is $[1\ 0]^T$ when agent i chooses A at t and is $\mathbf{1} - [1\ 0]^T = [0\ 1]^T$ otherwise, with $\mathbf{1} = [1\ 1]^T$. Let $x^A(t)$ denote the ratio of the A-players in the population at time t, and define the average population vector $s_A(t) = \begin{bmatrix} x^A(t) & 1 - x^A(t) \end{bmatrix}^T$. Then agent i's payoff at time t against the average population is calculated by $s_i(t)^T \pi_i \, s_A(t)$ where π_i is a coordination-game

payoff matrix defined by

$$\pi_i = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} R_i & S_i \\ T_i & P_i \end{pmatrix} \end{array}, \qquad R_i > T_i \quad \text{and} \quad P_i > S_i.$$

and the payoffs R_i, T_i, S_i and P_i correspond to strategy pairs A-against-A, B-against-A, A-against-B and B-against-B respectively. At each time step t, a random agent i is activated to revise her strategy according to the (myopic) best-response update rule dictating that agent i chooses that strategy which maximizes her payoff against the average population. In case both options A and B earn the same payoff, the agent sticks to her current strategy. Then the choice of agent i at time $t+1$ can be shown to bd

$$d_i(t+1) = \begin{cases} A & x^A(t) > \frac{S_i - P_i}{T_i - R_i + S_i - P_i} \\ d_i(t) & x^A(t) = \frac{S_i - P_i}{T_i - R_i + S_i - P_i} \\ B & x^A(t) < \frac{S_i - P_i}{T_i - R_i + S_i - P_i} \end{cases}.$$

Hence, the comparison of the coefficient $\frac{S_i - P_i}{T_i - R_i + S_i - P_i}$ with the ratio of A-players determines whether agent i changes her strategy when activated. Indeed, the coefficient $\frac{S_i - P_i}{T_i - R_i + S_i - P_i}$ plays the role of τ_i in the threshold model, which clarifies the equivalence of the above game theoretical model with the threshold model. We use the threshold model in the rest of the chapter.

7.3 Equilibrium states

To better understand type j and j^* agents, we first show how they revise their strategies according to the update rule (7.1). Denote the total number of A-playing agents in the population at time t by $n^A(t)$.

Lemma 7.1. *If agent $i \in \{1, \dots, n\}$ is a type j^* agent, $j \in \{1, \dots, l\}$, then the update rule* (7.1) *is equivalent to*

$$d_i(t+1) = \begin{cases} A & n^A(t) > n_j^* \\ d_i(t) & n^A(t) = n_j^* \\ B & n^A(t) < n_j^* \end{cases}.$$

If agent $i \in \{1, \dots, n\}$ is a type j agent, $j \in \{1, \dots, l\}$, then the update rule (7.1) *is equivalent to*

$$d_i(t+1) = \begin{cases} A & n^A(t) \geq n_j^* \\ B & n^A(t) < n_j^* \end{cases}.$$

Proof. The proof follows from (7.1) and the definitions of type j and j^* agents. □

Next, we characterize the equilibrium states. A state x^* is an *equilibrium state* (of the population dynamics), if whenever the state $x(t)$ equals x^*, it remains there afterwards, regardless of the

activation sequence. For $i = 0, 1, \ldots, l$, define the following $2l$-dimensional row vectors

$$s_i := (n_1, \tilde{n}_1, \ldots, n_i, \tilde{n}_i, 0, \ldots, 0),$$
$$s_i^* := \Big(n_1, \tilde{n}_1, \ldots, n_i, n_i^* - \sum_{j=1}^{i-1}(n_j + \tilde{n}_j) - n_i, 0, \ldots, 0\Big)$$

where s_0 and s_0^* are $1 \times 2l$ zero vector. All these vectors satisfy two necessary conditions for being an equilibrium state. First, from the update rule (7.1), we know that an agent plays A at an equilibrium if another agent with a smaller threshold also plays A; similarly, an agent would play B at an equilibrium if a greater-threshold agent plays B as well. Formally, we have the following result. Denote by $\mathcal{X}^*$ the set of all equilibrium states of the population dynamics governed by the update rule (7.1).

Lemma 7.2. *At any $x \in \mathcal{X}^*$, it holds that if $n_i^A > 0$ for some $i \in \{1, \ldots, n\}$, then $n_j^A = n_j$ and $\tilde{n}_j^A = \tilde{n}_j$ for all $j = 1, 2, \ldots, i-1$. Moreover, if $n_i^A = 0$ while $n_i \geq 1$ for some $i \in \{1, \ldots, n\}$, then $n_j^A = \tilde{n}_j^A = 0$ for all $j = i, i+1, \ldots, l$.*

Second, it follows from Lemma 7.1 that at an equilibrium, for every $j \in \{1, \ldots, l\}$, either n^A is no less than n_j^*, and hence, all type j agents play A, or n^A is less than n_j^*, and hence, all type j agents play B.

Lemma 7.3. *At any $x \in \mathcal{X}^*$, it holds that for every $i \in \{1, \ldots, n\}$, either $n_i^A = n_i$ or $n_i^A = 0$.*

Fulfilling Lemmas 7.2 and 7.3, s_i and s_i^*, $i = 1, 2, \ldots, l$, are candidates for being equilibrium states. We now show that they are the only candidates.

Lemma 7.4. $\mathcal{X}^* \subseteq \{s_i, s_i^*\}_{i=0}^l$.

Proof. Consider an equilibrium state $x^* \in \mathcal{X}^*$. If $x^* = s_0 = s_0^*$, the result is trivial; otherwise, in view of Lemma 7.3, the smallest index for which $n_j^A = n_j, j \in \{1, \ldots, l\}$ exists, which we denote by i. Then according to Lemmas 7.2 and 7.3, x^* has to take the following form

$$x^* = (n_1, \tilde{n}_1, \ldots, n_{i-1}, \tilde{n}_{i-1}, n_i, \tilde{n}_i^A, 0, 0, \ldots, 0).$$

Now the equality $n_i^A = n_i$ implies that all type i agents have chosen A, yielding $n^A \geq n_i^*$ in view of Lemma 7.1. If $n^A > n_i^*$, then Lemma 7.1 implies that all type i^* agents have chosen A as well, resulting in $x^* = s_i$. If on the other hand, $n^A = n_i^*$, then Lemma 7.1 implies that the number of type i^* agents who have chosen A are such that the total number of A-players becomes n_i^*, i.e., $\tilde{n}_i^A = n_i^* - \sum_{j=1}^{i-1}(n_j + \tilde{n}_j) - n_i$, resulting in $x^* = s_i^*$. Hence, in either case, $x^* \in \{s_i, s_i^*\}_{i=0}^l$. □

Now we determine those indices of s_i and s_i^* corresponding to the actual equilibrium states by using the sets $\mathcal{E} \subseteq \{0, \ldots, l\}$ defined by

$$\mathcal{E} = \{0, l\} \cup \Big\{i \in \{1, \ldots, l-1\} \,\Big|\, n_i^* \leq \sum_{j=1}^{i}(n_j + \tilde{n}_j) < n_{i+1}^*\Big\} \tag{7.3}$$

and consequently $\mathcal{E}^* \subseteq \mathcal{E} - \{0\}$ defined by

$$\mathcal{E}^* = \Big\{ i \in \{1, \dots, l\} \cap \mathcal{E} \;\Big|\; \sum_{j=1}^{i-1} (n_j + \tilde{n}_j) + n_i < n_i^* \Big\}. \tag{7.4}$$

Similar to the definitions in Chapter 6, we refer to the vectors $s_r, r \in \mathcal{E}$, and $s_r^*, r \in \mathcal{E}^*$, as *clean-cut* and *ruffled* states, respectively. First we show that these vectors are indeed a state of the system, i.e., they belong to the state space, and then we show that they are equilibrium states.

Lemma 7.5. *$s_r \in \mathcal{X}$ for $r \in \mathcal{E}$ and $s_r^* \in \mathcal{X}$ for $r \in \mathcal{E}^*$.*

Proof. The result is trivial for $s_r, r \in \mathcal{E}$, so we proceed to $s_r^*, r \in \mathcal{E}^*$. It suffices to show $0 \leq [s_r^*]_{2i-1} \leq n_i$ and $0 \leq [s_r^*]_{2i} \leq \tilde{n}_i$ for $i = 1, \dots, l$ where $[x]_i$ denotes the ith entry of x. According to the structure of s_r^*, we only need to show that $0 \leq [s_r^*]_{2r} \leq \tilde{n}_i$, i.e.,

$$0 \leq n_r^* - \sum_{j=1}^{r-1} (n_j + \tilde{n}_j) - n_r \leq \tilde{n}_r.$$

The second inequality is an immediate result of the left-hand-side inequality in the definition in (7.3). On the other hand, the first inequality follows directly from the inequality in (7.4). Thus, $s_r^* \in \mathcal{X}$, completing the proof. □

Lemma 7.6. *It holds that $\{s_r\}_{r \in \mathcal{E}} \cup \{s_r^*\}_{r \in \mathcal{E}^*} \subseteq \mathcal{X}^*$.*

Proof. The result is first shown for $s_r, r \in \mathcal{E}$. It can be easily verified that s_0 and s_l are equilibrium states. So let $r \in \mathcal{E} - \{0, l\}$. It suffices to show that for any $t_r \geq 0$,

$$x(t_r) = s_r \quad \Rightarrow \quad x(t_r + 1) = s_r.$$

We first observe that when $x(t_r) = s_r$ for some t_r, according to the definition of s_r and in view of (7.3),

$$n^A(t_r) = \sum_{j=1}^{r} (n_j + \tilde{n}_j) \geq n_r^* \overset{(7.2)}{\Longrightarrow} n^A(t_r) \geq n_i^* \quad \forall i \leq r.$$

Hence, from Lemma 7.1, for $i = 1, \dots, r$, if a type i or i^* agent is activated at t_r, she either chooses A or keeps her choice for time $t_r + 1$. On the other hand, according to the structure of s_r, the choices of all type i and type i^* agents are already A at t_r. Hence,

$$\begin{cases} n_i^A(t_r + 1) = n_i^A(t_r) = n_i \\ \tilde{n}_i^A(t_r + 1) = \tilde{n}_i^A(t_r) = \tilde{n}_i \end{cases} \quad \forall i \leq r. \tag{7.5}$$

Similarly it can be shown that

$$\begin{cases} n_i^A(t_r + 1) = n_i^A(t_r) = 0 \\ \tilde{n}_i^A(t_r + 1) = \tilde{n}_i^A(t_r) = 0 \end{cases} \quad \forall i > r. \tag{7.6}$$

Now (7.5) and (7.6) result in $x(t_r+1)=x(t_r)=s_r$, proving that s_r is an equilibrium state.

Next we show the result for $s_r^*, r\in\mathcal{E}^*$. It suffices to show that for any $t_r\geq 0$,

$$x(t_r)=s_r^* \quad \Rightarrow \quad x(t_r+1)=s_r^*.$$

We first observe that when $x(t_r)=s_r^*$ for some t_r, according to the definition of s_r^* and in view of (7.4),

$$n^A(t_r)=n_r^* \tag{7.7}$$

Hence, from Lemma 7.1, if a type r^* agent is activated at t_r, she keeps her strategy for time t_r+1, yielding

$$\tilde{n}_r^A(t_r+1)=\tilde{n}_r^A(t_r). \tag{7.8}$$

Moreover, again from Lemma 7.1, if a type r agent is activated at t_r, she chooses A. On the other hand, according to the structure of s_r^*, the choices of every type r agent are already A at time t_r. Hence,

$$n_r^A(t_r+1)=n_r^A(t_r). \tag{7.9}$$

Now due to (7.2), the following can be inferred from (7.7):

$$n^A(t_r)>n_i^* \quad \forall i<r \quad \text{ and } \quad n^A(t_r)<n_i^* \quad \forall i>r.$$

Hence, similar to (7.5) and (7.6), the following can be concluded

$$\begin{cases} n_i^A(t_r+1)=n_i^A(t_r) \\ \tilde{n}_i^A(t_r+1)=\tilde{n}_i^A(t_r) \end{cases} \qquad \forall i\neq r.$$

The above equation together with (7.8) and (7.9) imply that $x(t_r+1)=x(t_r)$. Hence, s_r^* is an equilibrium state, which completes the proof. □

Lemma 7.6 implies that the population under study is may have several equilibrium states depending on the types' population shares or equivalently the distribution of the thresholds among the agents. While the lemma only states that the clean-cut and ruffled states are equilibrium states, we show in the following that these are actually the only equilibrium states of the system. Let $\bar{\mathcal{E}}=\{1,\dots,l\}-\mathcal{E}$ and $\bar{\mathcal{E}}^*=\{1,\dots,l\}-\mathcal{E}^*$.

Lemma 7.7. *It holds that* $(\{s_r\}_{r\in\bar{\mathcal{E}}}\cup\{s_r^*\}_{r\in\bar{\mathcal{E}}^*})\cap\mathcal{X}^*=\emptyset$.

Proof. Let $r\in\bar{\mathcal{E}}$. Consider some time t_1 such that $x(t_1)=s_r$. Then by definition, $n^A(t_1)=\sum_{j=1}^{r}(n_j+\tilde{n}_j)$. Hence, in view of (7.3), $r\notin\mathcal{E}$ implies that one of the following is in force: (i) $n^A(t_1)<n_r^*$ or (ii) $n^A(t_1)\geq n_{r+1}^*$. Consider Case (i). If s_r is an equilibrium state, then $x(t)$ must equal s_r for all $t\geq t_1$ under any activation sequence. Consider some activation sequence, under which a type r agent is activated at some time $t_2>t_1$. If any agent has changed her choice between t_1 and t_2, then s_r is not an equilibrium state. So assume otherwise. Then $n^A(t_2)<n_r^*$. Hence, in view of Lemma 7.1, the type-r active agent updates to B at t_2+1. On the other hand, since $x(t_2)=s_r$, the choice of the type r agent is A at t_2. Hence, $x(t_2+1)\neq s_r$, implying that s_r is not

an equilibrium state. Similarly for Case (ii), a type $r+1$ agent can be shown to change her choice after some time, which again results in s_r not being an equilibrium state.

Now let $r \in \bar{\mathcal{E}}^*$. Consider some time t_1 such that $x(t_1) = s_r^*$. Then by definition, $n^A(t_1) = n_r^*$. Hence, according to (7.4), $r \notin \mathcal{E}^*$ yields $\sum_{j=1}^{r-1}(n_j + \tilde{n}_j) + n_r \geq n_r^*$. But then s_r^* is not even a state since $[s_r^*]_{2r} < 0$ (see the discussion in the proof of Lemma 7.6). This completes the proof. □

Theorem 7.1. *It holds that* $\mathcal{X}^* = \{s_r\}_{r\in\mathcal{E}} \cup \{s_r^*\}_{r\in\mathcal{E}^*}$.

Proof. Lemmas 7.4 and 7.7 result in $\mathcal{X}^* \subseteq \{s_r\}_{r\in\mathcal{E}} \cup \{s_r^*\}_{r\in\mathcal{E}^*}$. The proof then follows Lemma 7.6. □

Knowing the equilibrium states, now we are ready to proceed to the long-run behavior of the population dynamics.

7.4 Convergence analysis

We determine the asymptotic behavior of the state $x(t)$ for any given initial condition $x(0)$. The number of initially A-playing agents plays a key role in this procedure, which is illustrated in the following. Intuitively, if in a population, the ratio of the initial A-players with thresholds not greater than that of a particular type r^* agent exceeds the threshold of that agent, then both the type r^* agent and others with non-greater thresholds will eventually play A since they make enough A-players for themselves to update to A. We denote the maximum of r by y_1 that is formally defined by

$$y_1 \triangleq \max\Big\{r \in \mathcal{L} \,\Big|\, \sum_{i=1}^{r} n_i^A(0) + \tilde{n}_i^A(0) > n_r^*\Big\},$$

where $\mathcal{L} = \{0, 1, \ldots, l\}$, $n_0^* = -1$ and $\sum_{i=a}^{b}$ is defined to be zero if $a > b$. When the population reaches a state where all type j and j^* agents, $j \leq y_1$, have chosen A, then this state can be thought of as a new initial state, at which again some type j and j^* agents may eventually choose A. Let k_1 denote the greatest i, for which all type j and j^* agents, $j = 1, \ldots, i$, will eventually choose A, regardless of the activation sequence. Then as we show in Proposition 7.1, k_1 can be calculated by

$$k_1 \triangleq \begin{cases} y_1 & \text{if } \sum_{i=1}^{y_1} n_i + \tilde{n}_i \leq n_{y_1+1}^* \\ w & \text{otherwise} \end{cases} \tag{7.10}$$

where

$$w = \Big\{s \in \mathcal{L} \,\Big|\, \sum_{i=1}^{j} n_i + \tilde{n}_i > n_{j+1}^* \quad \forall j \in \{y_1, \ldots, s-1\}\Big\}.$$

In what follows, "almost surely" means "with probability one".

Proposition 7.1. *If for a given initial condition $x(0)$, $y_1 \geq 1$, then almost surely there exists some time T such that for $i = 1, \ldots, k_1$,*

$$n_i^A(t) = n_i, \quad \tilde{n}_i^A(t) = \tilde{n}_i \qquad \forall t \geq T. \tag{7.11}$$

Proof. First we prove (7.11) for $i = 1, \ldots, y_1$. For this we first show by contradiction that for $i = 1, \ldots, y_1$,

$$n_i^A(t) \geq n_i^A(0), \quad \tilde{n}_i^A(t) \geq \tilde{n}_i^A(0) \qquad \forall t \geq 0. \tag{7.12}$$

Assume on the contrary, there exists some time $t_1 > 0$ such that at least one of the inequalities in (7.12) fails. Let $t_2 \geq t_1$ be the first time this happens, namely, first of all, for $i = 1, \ldots, y_1$:

$$n_i^A(t) \geq n_i^A(0), \quad \tilde{n}_i^A(t) \geq \tilde{n}_i^A(0) \qquad \forall t < t_2, \tag{7.13}$$

and second, there exists some $q \in \{1, \ldots, y_1\}$ such that

$$n_q^A(t_2) < n_q^A(0) \quad \text{or} \quad \tilde{n}_q^A(t_2) < \tilde{n}_q^A(0).$$

Hence, a type q or q^* agent is activated at $t_2 - 1$ and changes her choice from A to B at t_2. On the other hand,

$$\begin{aligned} n^A(t_2 - 1) &\geq \sum_{j=1}^{y_1} n_j^A(t_2 - 1) + \tilde{n}_j^A(t_2 - 1) \\ &\overset{(7.13)}{\geq} \sum_{j=1}^{y_1} n_j^A(0) + \tilde{n}_j^A(0) > n_{y_1}^* \end{aligned}$$

where the last inequality follows from the definition of y_1. Hence, in view of (7.2), $n^A(t_2 - 1) > n_q^*$. So according to Lemma 7.1, if a type q or a type q^* agent is activated at $t_2 - 1$, she chooses A, not B, a contrary, implying that (7.12) is in force.

Next, observe that for all $t \geq 0$,

$$\begin{aligned} n^A(t) \geq \sum_{j=1}^{y_1} n_j^A(t) + \tilde{n}_j^A(t) &\overset{(7.12)}{\geq} \sum_{j=1}^{y_1} n_j^A(0) + \tilde{n}_j^A(0) > n_{y_1}^* \\ &\overset{(7.2)}{\Longrightarrow} n^A(t) > n_i^* \quad \forall i \leq y_1. \end{aligned}$$

Hence, in view of Lemma 7.1, for $i = 1, \ldots, y_1$, whenever a type i or i^* agent is activated, she chooses A. On the other hand, due to the persistent activation assumption, almost surely there exists some time $t_{y_1} \geq 0$ such that all type i and i^* agents are activated at least once before t_{y_1}, and hence, choose A before t_{y_1}. Therefore, (7.11) holds for $i = 1, \ldots, y_1$ and $\tau = t_{y_1}$.

Now if $k_1 = y_1$, the proof of the proposition is complete, so consider the case when $k_1 > y_1$ (the case $k_1 < y_1$ does not take place by the definition of k_1). We use strong induction to show (7.11) for $i = y_1, y_1 + 1 \ldots, k_1$. The statement was already shown for $i = y_1$. So assume that it holds for all $i \leq r$ where $y_1 \leq r \leq k_1 - 1$. Hence, almost surely there exists some time t_r such that (7.11) is in force for $i = r$ and $\tau = t_r$. Then

$$n^A(t) \geq \sum_{j=1}^{r} n_j^A(t) + \tilde{n}_j^A(t) = \sum_{j=1}^{r} n_j + \tilde{n}_j \qquad \forall t \geq t_r. \tag{7.14}$$

On the other hand, from (7.10), $k_1 > y_1$ implies $k_1 = w$. So $y_1 \leq r < w$. Therefore, based on the definition of w,

$$\sum_{j=1}^{r} n_j^A(t) + \tilde{n}_j^A(t) > n_{r+1}^* \overset{(7.14)}{\Longrightarrow} n^A(t) > n_{r+1}^* \qquad \forall t \geq t_r.$$

Then similar to what was shown above for $i = 1, \ldots, y_1$, the induction statement can be proven for $i = r + 1$. This completes the proof of the induction and hence the whole proposition. □

When the population reaches the state where all type j and j^* agents, $j = 1 \ldots, k_1$, have chosen A, still some other types may choose A, but this time it depends on the activation sequence. To encompass all these potential long-run-A-playing types, we define $v_{k_1} = 0$ if $k_1 = l$; otherwise, we define v_p, $p = k_1, \ldots, l-1$, by

$$v_p \triangleq \begin{cases} 1 & \text{if } \sum_{j=1}^{p} n_j + \tilde{n}_j + \sum_{j=p+1}^{l} n_j^A(0) + \tilde{n}_j^A(0) > n_{p+1}^* \\ 0 & \text{otherwise} \end{cases}$$

and then k_l by

$$k_l \triangleq \begin{cases} k_1 & \text{if } v_{k_1} = 0 \\ 1 + \min\{p \,|\, v_p = 1, v_{p+1} = 0\} & \text{otherwise} \end{cases}$$

where v_{l+1} is defined to be zero. Namely, for each p, we set $v_p = 1$ if the portion of individuals with thresholds not greater than that of a type p^* agent and others who initially play A, exceeds the threshold of a type $(p+1)^*$ agent; consequently, all type $p+1$ and $(p+1)^*$ agents may choose A in the long run if they update their choices before any other type or if in the meanwhile other initially A-playing individuals do not switch to B. Then intuitively, k_l provides the greatest possible type of agents, all individuals with thresholds non-greater than the threshold of which may eventually choose A. So we do not expect any of the other type j and j^* agents, $j = k_l + 1, \ldots, l$, to all choose A in the long run. Indeed one can show somewhat the opposite: all type j and j^* agents, $j = k_l + 2, \ldots, l$, will eventually choose B.

Proposition 7.2. *If for a given initial condition $x(0)$, $k_l \leq l - 2$, then almost surely there exists some time T such that for $i = k_l + 2, \ldots, l$,*

$$n_i^A(t) = \tilde{n}_i^A(t) = 0 \qquad \forall t \geq T. \tag{7.15}$$

Proof. We first show by contradiction that for $i = k_l + 2, \ldots, l$,

$$n_i^A(t) \leq n_i^A(0), \quad \tilde{n}_i^A(t) \leq \tilde{n}_i^A(0) \qquad \forall t \geq 0. \tag{7.16}$$

Assume on the contrary, there exists some time step $t_1 > 0$ such that at least one of the inequalities in (7.16) fails. Let $t_2 \geq t_1$ be the first time this happens, namely, first of all, for $i = k_l + 2, \ldots, l$:

$$n_i^A(t) \leq n_i^A(0), \quad \tilde{n}_i^A(t) \leq \tilde{n}_i^A(0) \qquad \forall t < t_2, \tag{7.17}$$

and second, there exists some $q \in \{k_l + 2, \ldots, l\}$ such that

$$n_q^A(t_2) > n_q^A(0) \quad \text{or} \quad \tilde{n}_q^A(t_2) > \tilde{n}_q^A(0).$$

Hence, a type q or q^* agent is activated at t_2-1 and changes her choice from B to A at time t_2. On the other hand,

$$\begin{aligned} n^A(t_2-1) &\leq \sum_{i=1}^{k_l} n_i + \tilde{n}_i + \sum_{i=k_l+1}^{l} n_i^A(t_2-1) + \tilde{n}_i^A(t_2-1) \\ &\overset{(7.17)}{\leq} \sum_{i=1}^{k_l} n_i + \tilde{n}_i + \sum_{i=k_l+1}^{l} n_i^A(0) + \tilde{n}_i^A(0) \\ &\leq n^*_{k_l+1} \end{aligned} \tag{7.18}$$

where the last inequality follows from the definitions of k_l and v_p. Now in view of (7.2), we can conclude from (7.18) that

$$n^A(t_2-1) < n_i^* \qquad \forall i \in \{k_l+2,\dots,l\}.$$

Hence, $n^A(t_2-1) < n_q^*$. Therefore, according to Lemma 7.1, if a type q or q^* agent is activated at t_2-1, she chooses B, not A, a contrary, implying that (7.16) is in force. The rest of the proof can be done similarly to that of Proposition 7.1. □

So far we know that all type j and j^* agents, $j=1,\dots,k_1$, almost surely will and all type i and i^* agents, $i=k_1+1,\dots,k_l$ may eventually choose A and all type j and j^* agents, $j=k_l+1,\dots,l$, almost surely will eventually choose B. Yet, it remains unknown whether $x(t)$ converges to a single equilibrium state or for example fluctuates between several states or converges to a limit cycle. As the main result of this chapter, we answer this question in the following theorem. Define the set $\bar{\mathcal{X}}$ by

$$\bar{\mathcal{X}} \triangleq \{s_r\}_{r\in\mathcal{K}} \cup \{s_r^*\}_{r\in\mathcal{K}^*}$$

where $\mathcal{K} = \{k_1, k_1+1,\dots,k_l\}\cap\mathcal{E}$ and $\mathcal{K}^* = \emptyset$ if $k_1 = l$, and $\mathcal{K}^* = \{k_1^*, k_1^*+1,\dots,k_l^*\}\cap\mathcal{E}^*$ otherwise, where $k_1^*, k_l^* \in \{1,\dots,l\}$ are defined by

$$k_1^* \triangleq \begin{cases} l+1 & k_1 = l \\ k_1+1 & k_1 < l \end{cases}, \qquad k_l^* \triangleq \begin{cases} l & k_l = l \\ k_l+1 & k_l < l \end{cases}.$$

Theorem 7.2. *Given any initial condition $x(0)$ and under the update rule (7.1), almost surely there exist some time T and some $x^* \in \bar{\mathcal{X}}$ such that*

$$x(t) = x^* \qquad \forall t \geq T. \tag{7.19}$$

The following lemma is the main step of the proof.

Lemma 7.8. *For each $p \in \{1,\dots,l\}$, almost surely there exists some time step t_p such that one of the following cases takes place for all $t \geq t_p$:*

1. *for $i=1,\dots,p$, $n_i^A(t) = n_i$ and $\tilde{n}_i^A(t) = \tilde{n}_i$;*
2. • *for $i=1,\dots,p-1$, $n_i^A(t) = n_i$ and $\tilde{n}_i^A(t) = \tilde{n}_i$,*

- *for* $i = p+1, \ldots, l$, $n_i^A(t) = \tilde{n}_i^A(t) = 0$,
- *and* $n_p^A(t) = n_p$ *and* $\tilde{n}_p^A(t) = n_p^* - \sum_{j=1}^{p-1}(n_j + \tilde{n}_j) - n_p < \tilde{n}_p$;

3. *for* $i = p, \ldots, l$, $n_i^A(t) = \tilde{n}_i^A(t) = 0$.

Proof. We prove by induction. First we need to show the result for $p = 1$; however, the proof for this case is similar to the general case $p = r \geq 1$. So we proceed as follows. Assume that the lemma is true for some $r \in \{1, \ldots, l-1\}$, implying that one of the three cases in the lemma takes place. Cases 2 and 3, however, are straightforward to prove since they already imply that for $i = r+1, \ldots, l$,

$$n_i^A(t) = \tilde{n}_i^A(t) = 0 \quad \forall t \geq t_r, \tag{7.20}$$

which matches the third case of the lemma for $p = r+1$ and $t_{r+1} = t_r$. So consider the situation when Case 1 holds for $p = r$, i.e.,

$$n_i^A(t) = n_i, \quad \tilde{n}_i^A(t) = \tilde{n}_i \quad \forall t \geq t_r \quad \forall i \leq r. \tag{7.21}$$

In general, one of the following two scenarios happens:

Scenario A: There exists some time $t_1 > t_r$*, at which a type* $r+1$ *agent is activated and switches to* B *at* $t_1 + 1$. Hence, in view of Lemma 7.1,

$$n^A(t_1) < n_{r+1}^* \overset{(7.2)}{\Longrightarrow} n^A(t_1) < n_i^* \quad \forall i \geq r+1. \tag{7.22}$$

Now we use contradiction to prove the following:

$$n^A(t) < n_i^* \quad \forall t \geq t_1 \quad \forall i \geq r+1. \tag{7.23}$$

Assume on the contrary that there exists some time when n^A becomes no less than n_i^* for some $i \geq r+1$. Let $t_3 \in (t_1, t_2]$ denote the first time this happens, i.e.,

$$n^A(t) < n_i^* \quad \forall t \leq t_3 - 1 \quad \forall i \geq r+1, \tag{7.24}$$

and

$$\exists q \in \{r+1, \ldots, l\} : n^A(t_3) = n_q^*.$$

Hence, an agent has changed her choice to A at t_3. However, none of the type i or i^* agent, $i \geq r+1$, can do that according to (7.24) and Lemma 7.1. The same holds for $i \leq r$ in view of (7.21), a contradiction, implying that (7.23) holds. Therefore, in view of Lemma 7.1 and due to the persistent activation assumption, almost surely there exists some time $t_{r+1} \geq t_1$ such that

$$n_i^A(t) = \tilde{n}_i^A(t) = 0 \quad \forall t \geq t_{r+1} \quad \forall i \geq r+1.$$

This matches Case 3 of the lemma for $p = r+1$. Hence, the induction statement holds for this scenario.

Scenario B: No type $r+1$ *agent switches to* B *after* t_r. First, we show by contradiction that

$$n^A(t) \geq n_{r+1}^* \quad \forall t \geq t_r. \tag{7.25}$$

Assume on the contrary that there exists some time $t_1 \geq t_r$ such that $n^A(t_1) < n^*_{r+1}$. Hence, in view of (7.2), for all $i \geq r+1$, $n^A(t_1) < n^*_i$. Then for $i \geq r+1$, if a type i or i^* agent is activated to update at t_1+1, she chooses B according to Lemma 7.1. On the other hand, for $i \leq r$, all type i and i^* agents have chosen A at t_r and do not change their choices afterwards. Hence, no agent switches to A at t_1+1, implying that $n^A(t_1+1) \leq n^A(t_1) < n^*_{r+1}$. Indeed by induction it can be shown that $n^A(t) < n^*_{r+1}$ for all $t \geq t_1$. So when a type $r+1$ agent is activated, she updates to B, a contradiction. Therefore, (7.25) is in force.

Hence, according to Lemma 7.1, almost surely there exists some time $t_2 \geq t_r$, before which all the type $r+1$ agents have updated their choices to A and do not change afterwards, i.e.,

$$n^A_{r+1}(t) = n_{r+1} \quad \forall t \geq t_2. \tag{7.26}$$

On the other hand, for the type $(r+1)^*$ agents, (7.25) and Lemma 7.1 imply that whenever a type $(r+1)^*$ agent is activated, she does not switch to B. So since the number of $(r+1)^*$ agents are finite, there exists some time $t_3 \geq t_2$ after which no $r+1^*$ agent switches her strategy. Then the number of A-playing agents of type $(r+1)^*$ does not change after t_3, resulting in one of the following cases:

Scenario B-1: there exists some time $t_3 \geq t_2$ such that

$$\tilde{n}^A_{r+1}(t) = \tilde{n}_{r+1} \quad \forall t \geq t_3.$$

Take $t_{r+1} = t_3$. Then according to (7.26) and (7.21), we arrive at Case 1 of the lemma for $p = r+1$. Hence, the induction statement holds for this part of Scenario B.

Scenario B-2: there exists some time $t_3 \geq t_2$ such that

$$\tilde{n}^A_{r+1}(t) = c < \tilde{n}_{r+1} \quad \forall t \geq t_3 \tag{7.27}$$

where $c \in \mathbb{Z}_{\geq 0}$ is constant. First we look at type i and i^* agents, $i = r+2, \ldots, l$. If $n^A(t) \geq n^*_{r+2}$ for all $t \geq t_3$, then in view of (7.2), $n^A(t) \geq n^*_{r+1}$ for all $t \geq t_3$. Hence, due to the persistent activation assumption, almost surely there exists some time t_s such that $\tilde{n}^A_{r+1}(t_s) = \tilde{n}_{r+1}$, which is not possible due to (7.27). So there exists some time $t_4 \geq t_3$, such that $n^A(t_4) < n^*_{r+2}$. Then similar to how (7.23) was shown, via contradiction it can be proven that

$$n^A(t) < n^*_{r+2} \quad \forall t \geq t_4.$$

Hence, in view of Lemma 7.1, almost surely there exists some time $t_5 \geq t_4$ such that

$$n^A_i(t) = \tilde{n}^A_i(t) = 0 \quad \forall t \geq t_5 \quad \forall i \geq r+2. \tag{7.28}$$

Now we look at type $(r+1)^*$ agents. almost surely there exists some time $t_6 \geq t_5$ when a $(r+1)^*$ agent is activated. The agent does not switch her strategy at t_6+1 since $t_6 \geq t_3$ and no $(r+1)^*$ agent switches her strategy after t_3. So from Lemma 7.1, $n^A(t_6) = n^*_{r+1}$. Hence, due to (7.21), (7.26) and (7.28), the following holds for $t = t_6$:

$$\tilde{n}^A_{r+1}(t) = n^*_{r+1} - \sum_{j=1}^{r+1-1} (n_j + \tilde{n}_j) - n_{r+1}. \tag{7.29}$$

On the other hand, according to (7.27), $\tilde{n}^A_{r+1}(t)$ is constant for $t \geq t_3$. Hence, (7.29) holds for all $t \geq t_3$. Then according to (7.21), (7.26), (7.28) and (7.29), we arrive at Case 2 of the lemma for $p = r + 1$.

So the induction statement holds for $r + 1$ in both Scenarios B-1 and B-2. Since it also holds in Scenario A, the proof is complete. □

Now we show that the population state almost surely reaches one of the clean-cut or ruffled equilibria. This enables us to complete the proof of Theorem 7.2 as follows.

Proposition 7.3. *Given any initial condition $x(0)$, almost surely there exist some time T and some final state $x^* \in \{s_r\}_{r\in\mathcal{E}} \cup \{s^*_r\}_{r\in\mathcal{E}^*}$ such that*

$$x(t) = x^* \quad \forall t \geq T.$$

Proof. The proof follows from Lemma 7.8. If Case 1 in Lemma 7.8 does not take place for $p = 1$, then either Case 2 or Case 3 happens, resulting in $x(t) = s^*_1$ or $x(t) = s_0$, respectively, for $t \geq t_p$. So consider the situation when Case 1 takes place for $p = 1$. Let r denote the greatest $p \in \{1, \ldots, l\}$, for which Case 1 takes place. If $r = l$, then $x(t) = s_l$ for $t \geq t_p$. Otherwise, for $p = r + 1$, either Case 2 or Case 3 takes place, resulting in $x(t) = s^*_{r+1}$ or $x(t) = s_r$, respectively, for $t \geq t_p$. To sum up, almost surely there exists some time T and some final state $x^* \in \{s_i\}^l_{i=1} \cup \{s^*_i\}^l_{i=1}$ such that $x(t) = x^*$ for all $t \geq T$. The proof is then complete in view of Lemma 7.7. □

Proof of Theorem 7.2. Proposition 7.1 confines x^* in Proposition 7.3 to $\{s_i\}^l_{i=k_1} \cup \{s^*_i\}^l_{i=k^*_1}$. Similarly Proposition 7.2 confines x^* to $\{s_i\}^{k_l}_{i=0} \cup \{s^*_i\}^{k^*_l}_{i=1}$. Hence, x^* is confined to $\{s_i\}^{k_l}_{i=k_1} \cup \{s^*_i\}^{k^*_l}_{i=k^*_1}$, which completes the proof. □

7.5 Stability analysis

From Theorem 7.2, we know that for any initial condition, the population dynamics almost surely reaches an equilibrium state. However, the stability of the equilibrium still remains an open problem that we investigate here. In what follows, by $\|\cdot\|$, we refer to the one-norm, i.e., $\|a\| = \sum_{j=1}^{2l} |a_j|$, and we denote a ball with the radius r and center s by $\mathcal{B}_r(s) = \{x \in \mathcal{X} \mid \|x-s\| < r\}$. By stability we refer to the standard definition of Lyapunov stability, but when all possible activation sequences are considered, i.e., an equilibrium state s is stable if for every ball $\mathcal{U}_\epsilon(s)$ there exists a ball $\mathcal{U}_\delta(s)$ such that for every $x(0) \in \mathcal{U}_\delta$, we have $x(t) \in \mathcal{U}_\epsilon$ for all $t \geq 0$ and for all persistent activation sequences. Since the state space $\mathcal{X}$ is discrete, ϵ and δ must be no less than 2; otherwise, $\mathcal{U}_\delta(s)$ for example, only contains the equilibrium state s, automatically implying $x(t) = s$ for all $t \geq 0$, even if s is unstable. Asymptotic stability is then defined correspondingly, namely s is asymptotically stable if it is stable and there exists some $\mathcal{U}_\delta$ such that if $x(0) \in \mathcal{U}_\delta$, then $\lim_{t\to\infty} \|x(t) - s\| = 0$.

First, we focus on the stability of the clean-cut equilibrium states. By definition, we know that $n^*_i \leq \sum^i_{j=1}(n_j + \tilde{n}_j) < n^*_{i+1}$ holds for each clean-cut equilibrium $s_i, i \in \mathcal{E}$. The following result states that each s_i is (asymptotically) stable if and only if the inequality is tightened.

Theorem 7.3. *The equilibrium state $s_i, i \in \mathcal{E}$, is stable if and only if*

$$n_i^* + 1 \leq \sum_{j=1}^{i} (n_j + \tilde{n}_j) < n_{i+1}^* - 1, \tag{7.30}$$

and is asymptotically stable if and only if

$$n_i^* + 1 < \sum_{j=1}^{i} (n_j + \tilde{n}_j) < n_{i+1}^* - 1 \tag{7.31}$$

where $n_0^* = -2$ and $n_{l+1}^* = n + 2$.

Now if the distribution of agents' thresholds is too diverse in a population, it is likely that $n_i^* + 1$ and $n_{i+1}^* - 1$ are close to each other for some $i \in \mathcal{E}$, making it unlikely for (7.30) to hold, implying the instability of s_i in view of the theorem.

For the proof of Theorem 7.3, we need to investigate the dynamics of $x(t)$ for those initial conditions with distance 1 from s_i that are captured by the following $2l$-dimensional vectors:

$$\begin{aligned} b_i(r) &= (n_1, \tilde{n}_1, \ldots, n_r - 1, \tilde{n}_r, \ldots, n_i, \tilde{n}_i, 0, \ldots, 0), \\ b_i^*(r) &= (n_1, \tilde{n}_1, \ldots, n_r, \tilde{n}_r - 1, \ldots, n_i, \tilde{n}_i, 0, \ldots, 0), \end{aligned}$$

where $1 \leq r \leq i$, and

$$\begin{aligned} c_i(r) &= (\underbrace{n_1, \tilde{n}_1, \ldots, n_i, \tilde{n}_i, 0, \ldots, 0}_{2r}, 1, 0, 0, \ldots, 0), \\ c_i^*(r) &= (\underbrace{n_1, \tilde{n}_1, \ldots, n_i, \tilde{n}_i, 0, \ldots, 0}_{2r}, 0, 1, 0, \ldots, 0), \end{aligned}$$

where $i \leq r \leq l - 1$. In what follows, by $x(0) = z$ where $z = b_i(r), b_i^*(r), c_i(r)$ or $c_i^*(r)$, we mean $x(0) = z \in \mathcal{X}$, namely z is a valid state. The following lemma describes the behavior of $x(t)$ when starting from any of the states $b_i(r)$ and $b_i^*(r)$, $1 \leq r \leq i$.

Lemma 7.9. *For an $s_i, i \in \mathcal{E} - \{0\}$, it holds that*

1. *if $\sum_{j=1}^{i} (n_j + \tilde{n}_j) - 1 > n_i^*$, then if $x(0) = b_i(r)$ or $x(0) = b_i^*(r)$, $1 \leq r \leq i$, almost surely there exists some time t_1 such that*

$$x(t) = x(0) \quad \forall t \leq t_1 \quad \text{and} \quad x(t) = s_i \quad \forall t > t_1.$$

2. *if $\sum_{j=1}^{i} (n_j + \tilde{n}_j) - 1 = n_i^*$, then*

   (a) *if $x(0) = b_i(r), 1 \leq r \leq i$, or $x(0) = b_i^*(r), 1 \leq r \leq i - 1$, almost surely there exists some time t_1 such that*

$$x(t) = x(0) \quad \forall t \leq t_1 \quad \text{and} \quad x(t) = s_i \quad \forall t > t_1.$$

(b) if $x(0) = b_i^*(i)$

$$x(t) = x(0) = s_i^* \quad \forall t.$$

3. if $\sum_{j=1}^{i}(n_j + \tilde{n}_j) - 1 < n_i^*$*, then*

(a) if $x(0) = b_i(r)$ *or* $x(0) = b_i^*(r)$, $1 \le r \le i-1$, *or if* $n_i + \tilde{n}_i \ge 2$ *and either* $x(0) = b_i(i)$ *or* $x(0) = b_i^*(i)$, *then there exists a nontrivial event of activation sequences, under which* $x(1) \notin \mathcal{B}_2(s_i)$.

(b) if $n_i + \tilde{n}_i = 1$, *then if* $x(0) = b_i(i)$ *or* $x(0) = b_i^*(i)$,

$$x(t) = x(0) = s_{i-1} \quad \forall t.$$

Proof. Case 1) If $x(0) = b_i(r)$ for some $r \le i$, then

$$n^A(0) = \sum_{j=1}^{l} n_j^A(0) + \tilde{n}_j^A(0) = \sum_{j=1}^{i}(n_j + \tilde{n}_j) - 1 > n_i^*. \tag{7.32}$$

On the other hand, according to the definition of s_i, it holds that $\sum_{j=1}^{i}(n_j + \tilde{n}_j) < n_{i+1}^*$. Hence,

$$n^A(0) = \sum_{j=1}^{i}(n_j + \tilde{n}_j) - 1 < n_{i+1}^*. \tag{7.33}$$

Now if the active agent at $t = 0$ is a type j or j^* agent, $j = 1, \ldots, i$, then in view of Lemma 7.1, (7.32) implies that she will update to A at $t = 1$. On the other hand, unless the active agent is a type r agent, she is already playing A at $t = 0$, and hence, does not change her strategy at $t = 1$, resulting in

$$x(1) = x(0). \tag{7.34}$$

Similarly if the active agent at $t = 0$ is a type j or j^* agent, $j = i + 1, \ldots, l$, then in view of Lemma 7.1, (7.33) implies that she will update to B at $t = 1$. Since all such agents are already playing B at $t = 0$, none will change their strategies at $t = 1$. Hence, (7.34) is also in force for this case. Indeed by induction it can be shown that as long as a type r agent is not activated, it holds that $x(t) = x(0)$. However, because of the persistent assumption on the activation sequence, we know that almost surely there exists some time t_1 at which a type r agent is activated. Then in view of Lemma 7.1 and (7.32), she updates her strategy to A in the next time step, implying $x(t_1 + 1) = s_i$. Since s_i is an equilibrium state, it follows that $x(t) = s_i$ for all $t \ge t_1 + 1$. The same can be shown for when $x(0) = b_i^*(r)$, which completes the proof of the first case.

Case 2) Part 2-a) can be proven similarly to Case 1), so we look at Part 2-b). If $\tilde{n}_i = 0$, then $b_i^*(i) \notin \mathcal{X}$, and hence, $x(0)$ cannot equal $b_i^*(i)$, making the result trivial. So consider the situation when $\tilde{n}_i \ge 1$. If $x(0) = b_i^*(i)$, then

$$n^A(0) = \sum_{j=1}^{i}(n_j + \tilde{n}_j) - 1 = n_i^* \tag{7.35}$$

Hence, $\tilde{n}_i = n_i^* - \sum_{j=1}^{i-1}(n_j + \tilde{n}_j) - n_i$, implying that $n_0^A = s_i^*$. On the other hand, from (7.2), $n_i^* < n_{i+1}^*$, resulting in $i \in \mathcal{E}$ in view of (7.3). Then $\tilde{n}_i > 0$ results in $i \in \mathcal{E}^*$ in view of (7.4), completing the proof of this case.

Case 3) If $x(0) = b_i(r)$ or $b_i^*(r)$ for some $r \leq i$, then

$$n^A(0) = \sum_{j=1}^{i}(n_j + \tilde{n}_j) - 1 < n_i^* \tag{7.36}$$

Now consider the following two cases:

Case 3-a) either $r \leq i-1$ or both $r = i$ and $n_i + \tilde{n}_i \geq 2$. Then there exists a type i or i^* agent whose choice is A at $t = 0$. Under a nontrivial event of activation sequences, this agent is activated at $t = 0$, and hence, will choose B at $t = 1$, resulting in $x(0) \notin \mathcal{D}^1$.

Case 3-b) $r = i$ and $n_i + \tilde{n}_i = 1$. Then $x(0) = s_{i-1}$. We show that $i - 1 \in \mathcal{E}$. Since $i \in \mathcal{E}$, we know that

$$n^A(0) = \sum_{j=1}^{i}(n_j + \tilde{n}_j) - 1 \geq n_i^* - 1,$$

which in view of (7.36) yields $n^A(0) = n_i^* - 1$. Hence,

$$n_{i-1}^* \leq n^A(0) = \sum_{j=1}^{i-1}(n_j + \tilde{n}_j) = n_i^* - 1 < n_i^*,$$

which implies $i-1 \in \mathcal{E}$ in view of (7.3), completing the proof of this case, and hence the whole. □

The following lemma describes the behavior of $x(t)$ when starting from the states $c_i(r)$ and $c_i^*(r)$, $i \leq r \leq l-1$.

Lemma 7.10. *For an $s_i, i \in \mathcal{E} - \{l\}$, it holds that*

1. *if $\sum_{j=1}^{i}(n_j + \tilde{n}_j) + 1 < n_{i+1}^*$, then if $x(0) = c_i(r)$ or $x(0) = c_i^*(r)$, $i \leq r \leq l-1$, almost surely there exists some time t_1 such that*

$$x(t) = x(0) \ \ \forall t \leq t_1 \quad \text{and} \quad x(t) = s_i \ \ \forall t > t_1.$$

2. *if $\sum_{j=1}^{i}(n_j + \tilde{n}_j) + 1 = n_{i+1}^*$, then*

   (a) *if $x(0) = c_i(r)$ or $x(0) = c_i^*(r), i+1 \leq r \leq l-1$, almost surely there exists some time t_1 such that*
   $$x(t) = x(0) \ \ \forall t \leq t_1 \quad \text{and} \quad x(t) = s_i \ \ \forall t > t_1.$$

   (b) *if $n_i \neq 0$ and $x(0) = c_i^*(i)$ or if $n_i \neq 1$ and $x(0) = c_i(i)$, there exists a nontrivial event of activation sequences, under which $x(1) \notin \mathcal{B}_2(s_i)$.*

   (c) *if $n_i = 0$ and $x(0) = c_i^*(i)$,*
   $$x(t) = x(0) = s_{i+1}^* \quad \forall t.$$

   (d) *if $n_i = 1$ and $x(0) = c_i(i)$*
   $$x(t) = x(0) = s_{i+1} \quad \forall t.$$

Proof. Case 1) If $x(0) = b_i(r)$ for some $r \leq i$, then

$$n^A(0) = \sum_{j=1}^{i}(n_j + \tilde{n}_j) + 1 < n^*_{i+1}. \tag{7.37}$$

On the other hand, according to the definition of s_i, it holds that $\sum_{j=1}^{i}(n_j + \tilde{n}_j) \geq n^*_i$. Hence,

$$n^A(0) = \sum_{j=1}^{i}(n_j + \tilde{n}_j) + 1 > n^*_i. \tag{7.38}$$

The rest of the proof of this case can be done similarly to that of Lemma 7.9, Case 1).

Case 2) The proof is similar to that of Cases 2) and 3) of Lemma 7.9. □

The case $\sum_{j=1}^{i}(n_j + \tilde{n}_j) + 1 > n^*_{i+1}$ is not investigated in the lemma since it never takes place. This is because from $i \in \mathcal{E}$, we know that $\sum_{j=1}^{i}(n_j + \tilde{n}_j) + 1 < n^*_{i+1} + 1$, which is in contradiction with the previous inequality. Knowing the behavior of $x(t)$ for all states with distance 1 from a clean-cut equilibrium state, we now proceed to the proof of the main result.

Proof of Theorem 7.3. We prove the result for stability; the result for asymptotic stability can be done similarly. (sufficiency) We show that $\delta = 2$ satisfies the stability condition for any $\mathcal{B}_\epsilon, \epsilon \geq 2$. So consider some $x(0) \in \mathcal{B}_2(s_i), x(0) \in \mathcal{X}$, implying $x(0) \in \mathcal{D}^2$. Should $x(0)$ be at the equilibrium state s_i, it would follow that $x(t) = s_i \in \mathcal{B}_2$ for all $t \geq 0$. So consider the second case, that is when $x(0)$ equals one of $b^*_i(r)$ and $b_i(r)$ for some $r \leq i$ or one of $c_i(r)$ and $c^*_i(r)$ for some $r \geq i$. Then in view of Lemmas 7.9 and 7.10, condition (7.31) results in again $x(t) = s_i \in \mathcal{B}_2$ for all $t \geq 0$, implying the stability of s_i.

(necessity) If s_i is stable, then for any initial condition $x(0) \in \mathcal{D}^2$, the smallest acceptable ball with center s_i, we must have $x(t) \in \mathcal{B}_2$. The proof then follows Lemmas 7.9 and 7.10. □

Now we focus on the stability of a ruffled equilibrium state, $s^*_i, i \in \mathcal{E}^*$, which we intuitively expect to satisfy a weaker notion of stability since it requires the total number of A-players to be exactly equal to a fixed value, i.e., n^*_i.

Theorem 7.4. *The equilibrium state $s^*_i, i \in \mathcal{E}^*$, is unstable.*

For the proof, we need to investigate initial conditions with distance 1 from s^*_i, which are captured by the following vectors:

$$\tilde{b}_i(r) = s^*_i - \mathbf{1}_{2r-1}, \qquad \tilde{b}^*_i(r) = s^*_i - \mathbf{1}_{2r}$$

where $1 \leq r \leq i$ and $\mathbf{1}_i$ is the ith row of the $2l \times 2l$ identity matrix, and

$$\tilde{c}_i(r) = s^*_i + \mathbf{1}_{2r+1}, \qquad \tilde{c}^*_i(r) = s^*_i + \mathbf{1}_{2r+2}$$

where $i \leq r \leq l - 1$. Since Theorem 7.4 postulates the instability of s^*_i, it suffices to show that for each $s^*_i, i \in \mathcal{E}^*$, there exists one of $\tilde{b}_i(r), \tilde{b}^*_i(r), \tilde{c}_i(r)$ and $\tilde{c}^*_i(r)$ that drives $x(t)$ to a state out of $\mathcal{B}_2(s^*_i)$, regardless of the activation sequence.

Lemma 7.11. *For an $s^*_i, i \in \mathcal{E}^*$, it holds that*

1. *if* $i \neq l$, *then if* $x(0) = \tilde{c}_i(r)$ *or* $x(0) = \tilde{c}_i^*(r)$, $i \leq r \leq l-1$, *there exists a nontrivial event of activation sequences, under which* $x(1) \notin \mathcal{B}_2(s_i^*)$.

2. *if* $i = l$ *and* $\tilde{n}_l \geq 2$, *then if* $x(0) = \tilde{b}_i^*(i)$, *there exists a nontrivial event of activation sequences, under which* $x(1) \notin \mathcal{B}_2(s_i^*)$.

3. *if* $i = l$, $\tilde{n}_l = 1$ *and* $n_l \geq 1$, *then if* $x(0) = \tilde{b}_i(i)$, *there exists a nontrivial event of activation sequences, under which* $x(1) \notin \mathcal{B}_2(s_i^*)$.

The case $i = 0$ is not investigated in the lemma since $0 \notin \mathcal{E}^*$. Moreover, the case $i = l, \tilde{n}_l = 1$ and $n_l = 0$ is not possible since then $i \in \mathcal{E}$ not $\mathcal{E}^*$.

Proof of Theorem 7.4: The proof can be done similarly to that of Theorem 7.3, but when Lemma 7.11 is used instead of Lemmas 7.9 and 7.10. □

So unlike clean-cut equilibria that may even be asymptotically stable, ruffled equilibrium states are unstable, implying that small perturbations from the equilibrium may lead to moving to some states.

7.6 Contagion probability

As discussed in Section 7.4, the agent activation sequence is random, so the population dynamics may reach different equilibrium states in the long run, under different activation sequences. In other words, given an initial condition, each state in $\bar{\mathcal{X}}$ may be reached by $x(t)$ with a certain probability. We are interested in the contagion probability, that is the probability of reaching the state s_l where every individual has chosen A. To simplify the analysis, we limit ourselves to a heterogeneous population with just two types: type 1 and 2, and no type 1^* or 2^*, i.e., $\tilde{n}_1 = \tilde{n}_2 = 0$. After all, rarely the threshold of an agent may exactly equal n_i^*/n for some $i = 1, \ldots, n$, so this is not a strong assumption. Then we find a recurrence equation for the probability of contagion as follows. Define the recursive function $P(\cdot, \cdot) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ by

$$P(a, b) = \frac{n_1 - a}{n_1 - a + b} P(a + 1, b) + \frac{b}{n_1 - a + b} P(a, b - 1), \tag{7.39}$$

for $a < n_1$, $b > 0$ and $n_1^* \leq a + b < n_2^*$, with the boundary conditions

$$P(a, b) = \begin{cases} 0 & a + b < n_1^* \\ 1 & a + b \geq n_2^* \\ 0 & b < n_2^* - n_1 \\ 0 & b = 0, n_1 < n_2^* \\ 1 & b = 0, a \geq n_1^*, n_1 \geq n_2^* \end{cases}. \tag{7.40}$$

Proposition 7.4. *Consider a heterogeneous population of type 1 and 2 agents. Given an initial condition* $x(0) = (n_1^A(0), 0, n_2^A(0), 0)$, *the contagion probability equals* $P(n_1^A(0), n_2^A(0))$.

Proof. We observe that the boundary conditions (7.40) are equivalent to

$$P(a,b) = \begin{cases} 0 & a+b < n_1^* \\ 1 & a+b \geq n_2^* \\ 0 & n_1^* \leq a+b < n_2^*, a = n_1 \\ 0 & n_1^* \leq a+b < n_2^*, b = 0, n_1 < n_2^* \\ 1 & n_1^* \leq a+b < n_2^*, b = 0, n_1 \geq n_2^* \end{cases} \tag{7.41}$$

which together with (7.39) result in a unique solution for $P(a,b)$ for each pair (a,b). Therefore, it suffices to show that the contagion probability of a population with initial conditions $x(0) = (a,0,b,0)$ satisfies the recurrence equation (7.39) and the boundary conditions (7.41). The boundary conditions can be easily verified. As for the recurrence equation, we know that the contagion probability for $x(0)$ equals the sum of the probabilities of different possible states taking place at $t = 1$, times the contagion probabilities from those new states $x(1)$. Namely, for $a < n_1$, $b > 0$ and $n_1^* \leq a+b < n_2^*$, one of the following may take place at $t = 0$ and $t = 1$:

1) The active agent at $t = 0$ is an A-playing type 1 or a B-playing type 2 agent, which occurs with probability $\frac{a+n_2-b}{n}$. Then the active agent does not change her choice, resulting in $n_1^A(1) = n_1^A(0), n_2^A(1) = n_2^A(0)$. So the contagion probability from $x(1)$ equals $P(a,b)$.

2) The active agent at $t = 0$ is a B-playing type 1 agent, which occurs with probability $\frac{n_1-a}{n}$. Then the active agent chooses A, resulting in $n_1^A(1) = n_1^A(0)+1, n_2^A(1) = n_2^A(0)$. So the contagion probability from $x(1)$ equals $P(a+1,b)$.

3) The active agent at $t = 0$ is an A-playing type 2 agent, which occurs with probability $\frac{b}{n}$. Then the active agent chooses B, resulting in $n_1^A(1) = n_1^A(0), n_2^A(1) = n_2^A(0)-1$. So the contagion probability from $x(1)$ equals $P(a,b-1)$.

Therefore, we acquire the recurrence equation

$$P(a,b) = \frac{a+n_2-b}{n}P(a,b) + \frac{n_1-a}{n}P(a+1,b) + \frac{b}{n}P(a,b-1),$$

which results in (7.39), completing the proof. □

Example 7.1. Consider a population of 100 individuals of types 1 and 2 with populations $n_1 = 20$ and $n_2 = 80$ and $n_1^* = 25$ and $n_2^* = 33$ respectively. Using Proposition 7.4 and by solving the recurrence equation (7.39), we obtain a contagion probability of $3.13e-8\%$ for an initial condition of 12 A-playing type 1 and 14 B-playing type 2 individuals, i.e., $P(12,14) = 3.13e-10$, and that of 6.7% for an initial condition of 19 A-playing type 1 and 14 B-playing type 2 individuals, i.e., $P(19,14) = .067$. In general, the expected probability of contagion based on the number of initial A-players in the population are summarized in Table 7.1 that is taken from [1]. As show in the table, the contagion probability for low number of initial A-players is quite low, but exponentially grows as the number increases.

$n^A(0)$	$\max P$	$\mathrm{Exp}[P]$
25	0	0
26	0.37	0.05
27	1.05	0.16
28	2.35	0.39
29	4.83	0.90
30	9.42	2
31	17.72	4.41
32	32.35	9.81
33	57.56	22.74
34	100	100

Table 7.1: **Maximum and expected contagion probabilities for different initial A-players.** The population consists of 20 type 1 and 80 type 2 individuals where $n_1^* = 25$ and $n_2^* = 33$. For each number of initial A-players in the population, all possible combinations of type 1 and type 2 individuals with different choices of A and B are considered and the corresponding contagion probabilities are calculated using the recurrence equation (7.39) and the boundary conditions (7.40). Then the maximum probability, $\max P$, and the expected (average) probability, $\mathrm{Exp}[P]$ are obtained.

7.7 Concluding remarks

We have performed convergence and stability analysis on a finite well-mixed heterogeneous population where each individual is associated with a possibly unique threshold, and the agents update asynchronously based on the linear threshold model. We have shown that for any given initial condition, almost surely after a finite time, the population reaches an equilibrium where no agent violates her threshold; however, the equilibrium is not necessarily uniquely determined by the initial condition, but depends on the activation sequence of the agents. Moreover, the dynamics may be associated with several stable equilibria, i.e., the clean-cuts, where both A-playing and B-playing agents coexist in a well-mixed population. Such coexistence is impossible in homogeneous populations, and thus results highlight the role of perception differences or, in general, the heterogeneity of the population in linear threshold models or equivalently the best-response dynamics. We are currently working together with theoretical biologists and sociologists to understand better the effect of heterogeneity using real data.

Part III

Finite Networked Populations

Chapter 8

Networks of conforming or non-conforming individuals tend to reach satisfactory decisions

Binary decisions of agents coupled in networks can often be classified into two types: *coordination*, where an agent takes an action if enough neighbors are using that action, as in the spread of social norms, innovations, and viral epidemics, and *anti-coordination*, where too many neighbors taking a particular action causes an agent to take the opposite action, as in traffic congestion, crowd dispersion, and division of labor. Both of these cases can be modeled using linear threshold-based dynamics, and a fundamental question is whether the individuals in such networks are likely to reach decisions with which they are satisfied. We show that in the coordination case, and perhaps more surprisingly, also in the anti-coordination case, the agents will indeed always tend to reach satisfactory decisions, i.e., the network will almost surely reach an equilibrium state. This holds for every network topology and every distribution of thresholds, for both asynchronous and partially synchronous decision-making updates. These results reveal that irregular network topology, population heterogeneity and partial synchrony are not sufficient to cause cycles or non-convergence in linear threshold dynamics; rather, other factors such as imitation or the coexistence of coordinating and anti-coordinating agents must play a role.

8.1 Introduction

After presenting the results on infinite and finite well-mixed populations in Parts I and II, now we investigate finite networked populations in this part, and start with populations governed by the best-response update rule in this chapter. The sharing of resources, division of labor, and dispersion of crowds represent a few examples of collective behaviors that can emerge on networks of interacting agents when the adoption of a particular action by too many individuals deters others from adopting that action. On the other hand, when individuals taking an action makes it more likely that others will adopt that action, behaviors such as the spread of social norms, technological innovations, and viral infections can occur. In either case, various factors are thought to contribute to the emergence of such collective behaviors, including network topology, interaction payoffs,

strategy-update rules, and population diversity [105, 50, 159]. In particular, when agents can take one of a finite number of states at a given time, whether by voluntary decisions or involuntary reactions, the fundamental questions include (i) whether the agents will eventually settle on a fixed state/action, (ii) which distribution of actions the network will converge to, and (iii) how long it will take for the network to converge. While most of the literature focuses on the second and especially third questions, particularly for homogeneous populations where individuals share the same utility function [187, 84, 9, 109], the first question is perhaps more fundamental since it deals with convergence at the agent level. Undoubtedly, this action settling is crucial to a wide range of decision making populations, from collective nest selection by ants [101] to stabilization of financial markets [30]. Moreover, without establishing such convergence, answers to questions (ii) and (iii) cannot be used to design policies at the local or neighborhood level [144], which may help to improve cost efficiency by targeting influential nodes in a complex network.

Best response dynamics is one of the most widely used models to study the types of problems mentioned above; this is due in part to its rational nature as well as its broad applicability. Indeed recent experimental studies have shown that humans often use myopic best responses when making decisions in social networks [100]. The idea behind best response dynamics is simple: in a network of interacting agents, each agent takes the action that results in the best cumulative outcome against its neighbors. This framework is frequently considered in a game theoretic context [149, 85, 139, 158], but its impact is further broadened by the fact that when there are two available actions for each agent, it is equivalent to a *linear threshold model* [81]. This comes with an alternative yet also intuitive interpretation: when a sufficient fraction of my neighbors are taking action A, I will also take action A. The reverse is also possible: if too many of my neighbors are taking action A, I will switch to action B. We call agents of the former type *coordinating* and the latter, *anti-coordinating*, according to the standard coordination and anti-coordination games, which are also sometimes referred to as games of strategic complements and strategic substitutes, respectively [37]. These switches may be considered voluntary as in changing an opinion, preference, or habit, or involuntary, as getting infected by virus or defaulting on a loan. In general, the thresholds of the agents need not be the same, and this is equivalent to agents having asymmetric payoffs in the corresponding matrix game. Depending on the application, this can represent differences in the willingness to follow a crowd, susceptibility to infection, perceived value of interactions, and many other individual characteristics [53]. Variation in thresholds can thus be thought of as heterogeneity of a population.

The linear threshold model was first motivated and discussed by Mark Granovetter in the context of fully-connected networks, with an example of how the incitement of riots in crowds depends critically on the distribution of individual thresholds that will cause each person to join the riot [53]. Since then, a rich literature has developed around questions related to stability and convergence of best-response dynamics in matrix games on well-mixed populations [136, 7]. Convergence of homogeneous well-mixed populations under more general non-matrix games has been established in [34] in the context of congestion games, using potential functions. The convergence properties of networks with arbitrary structure has also been investigated under various conditions, particularly in the homogeneous (symmetric) case [110, 88, 92]. Recently in [37], a combination of mean-field approximations and simulations were used to show that synchronous best-response dynamics in symmetric coordination or anti-coordination games tend to always converge to Nash equilibria. However, in the case of asymmetric coordination games, it was shown in

[2] that synchronous best-response dynamics may not converge to a single action but rather to a cycle of alternating actions. Symmetry thus seems to be a significant factor in the convergence of the threshold model, but we will show that asynchrony perhaps plays an even more important role, because regardless of the symmetry, the network always converges in the asynchronous case. In addition to making the convergence more likely, compared to synchronous models, asynchronous dynamics can provide a more realistic model of the time-line over which independent agents make decisions and receive information, and they are particularly suitable when the payoff dynamics can be thought of as fast compared to the update dynamics. For example, decisions on matters such as which product to buy, which political party to vote for, or which traffic route to take may be the culmination of many individual interactions. Nevertheless, our results do not require that only one agent can update at a time; in fact, we show that even small asynchronous perturbations to fully synchronous dynamics lead to equilibrium convergence.

In this chapter, we show that every network consisting of anti-coordinating agents with asynchronous best-response dynamics will eventually reach an equilibrium state, even if each agent has a different threshold. Moreover, we show that the same result holds for networks of coordinating agents. As a corollary, we establish the existence of pure-strategy Nash equilibria in both cases for arbitrary networks and arbitrary payoffs. On the question of convergence time, we show that in such networks, the total number of times an agent switches strategies is no greater than six times the number of edges in the network. In the case of partial synchrony, when a random number of agents can update simultaneously, we show that the network still almost surely reaches an equilibrium. It follows that irregular network topology, population heterogeneity and synchrony of decisions between two or more agents (as long as random asynchronous updating is not completely excluded) are not sufficient to cause non-convergence or cycles in best-response dynamics; rather, possible causes include the occasional use of non-best-response strategies, randomization, or a mixture of coordinating and anti-coordinating agents. Indeed we provide a small example demonstrating the possibility of cycles in networks containing both coordinating and anti-coordinating agents.

8.2 Asynchronous best response dynamics

Consider an undirected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the nodes $\mathcal{V} = \{1, \dots, n\}$ correspond to agents and each edge in the set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents a 2-player game between neighboring agents. Each agent $i \in \mathcal{V}$ chooses pure strategies from a binary set $\mathcal{S} := \{A, B\}$ and receives a payoff upon completion of game according to the matrix:

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \end{array}, \qquad a_i, b_i, c_i, d_i \in \mathbb{R}.$$

The dynamics take place over a sequence of discrete time $k = 0, 1, 2, \dots$. Let $x_i(k) \in \mathcal{S}$ denote the strategy of agent i at time k, and denote the number of neighbors of agent i playing A and B at time k by $n_i^A(k)$ and $n_i^B(k)$, respectively. When there is no ambiguity, we may sometimes omit the time k for compactness of notation. The total payoffs to each agent i at time k are

accumulated over all neighbors, and are therefore equal to $a_i n_i^A(k) + b_i n_i^B(k)$ when $x_i(k) = A$, or $c_i n_i^A(k) + d_i n_i^B(k)$ when $x_i(k) = B$.

In asynchronous (myopic) best-response dynamics, one agent at a time becomes active and chooses a single action to play against all neighbors. The active agent at time k updates at time $k+1$ to the strategy that achieves the highest total payoff, i.e. is the best response, against the strategies of its neighbors at time k:

$$x_i(k+1)=\begin{cases} A, & \text{if } a_i n_i^A + b_i n_i^B > c_i n_i^A + d_i n_i^B \\ B, & \text{if } a_i n_i^A + b_i n_i^B < c_i n_i^A + d_i n_i^B \\ z_i, & \text{if } a_i n_i^A + b_i n_i^B = c_i n_i^A + d_i n_i^B \end{cases}.$$

In the case that strategies A and B result in equal payoffs, both strategies are best responses and we use the notation z_i, which is defined to be A, B or $x_i(k)$, to allow for several possibilities for this equality case. Note that we do not require all agents to have the same z_i. That is, when both A and B are best responses, some agents may choose A, others may choose B, and others may keep their current strategy; however, the agents cannot change their choice of z_i over time.

It is convenient to rewrite these dynamics in terms of the number of neighbors playing each strategy. Let $\deg_i$ denote the total number of neighbors of agent i. We can simplify the conditions above by using the fact that $n_i^B = \deg_i - n_i^A$ and rearranging terms:

$$\begin{aligned} a_i n_i^A + b_i(\deg_i - n_i^A) &> c_i n_i^A + d_i(\deg_i - n_i^A) \\ n_i^A(a_i - c_i + d_i - b_i) &> \deg_i(d_i - b_i) \\ \delta_i n_i^A &> \gamma_i \deg_i, \end{aligned} \tag{8.1}$$

where $\delta_i := a_i - c_i + d_i - b_i$ and $\gamma_i := d_i - b_i$. The cases of '$<$' and '$=$' can be handled similarly. Consider the case when $\delta_i \neq 0$, and let $\tau_i := \frac{\gamma_i}{\delta_i}$ denote a *threshold* for agent i. Depending on the sign of δ_i, we have two possible types of best-response update rules. If $\delta_i > 0$, the update rule is given by:

$$x_i(k+1) = \begin{cases} A & \text{if } n_i^A(k) > \tau_i \deg_i \\ B & \text{if } n_i^A(k) < \tau_i \deg_i \\ z_i & \text{if } n_i^A(k) = \tau_i \deg_i \end{cases}. \tag{8.2}$$

These agents switch to strategy A if a sufficient fraction of neighbors are using that strategy, and likewise for strategy B. On the other hand, for $\delta_i < 0$, if a sufficient fraction of neighbors are playing A, the agent will switch to B, and vice versa. This update rule is given by:

$$x_i(k+1) = \begin{cases} A & \text{if } n_i^A(k) < \tau_i \deg_i \\ B & \text{if } n_i^A(k) > \tau_i \deg_i \\ z_i & \text{if } n_i^A(k) = \tau_i \deg_i \end{cases}. \tag{8.3}$$

In the case that $\tau_i \notin [0,1]$, it is straightforward to show that there exists a strictly dominant strategy, and the update rule (8.2) or (8.3) is equivalent to one in which $\tau_i \in \{0,1\}$ and $z_i \in \{A,B\}$. The same holds for when $\delta_i = 0$. Agents for which $\delta_i \geq 0$ are called *coordinating* and can be

modeled by (8.2). Agents for which $\delta_i \leq 0$ are called *anti-coordinating* and can be modeled by (8.3). Therefore, every agent can be described as a coordinating or an anti-coordinating agent (or both).

Let $\Gamma := (\mathcal{G}, \tau, \{+, -, \pm\})$ denote a *network game*, which consists of the network $\mathcal{G}$, a vector of agent thresholds $\tau = (\tau_1, \ldots, \tau_n)^\top$, and one of $+$, $-$, or $\pm$, corresponding to the cases of all coordinating, all anti-coordinating, or a mixture of both types of agents, respectively. The dynamics in (8.2) and (8.3) are in the form of the standard *linear threshold model* [53]. An equilibrium state in the threshold model is a state in which the number of A-neighbors of each agent does not violate the threshold that would cause them to change strategies. For example, in a network of anti-coordinating agents in which $z_i = B$ for all i, this means that for each agent $i \in \mathcal{V}$, $x_i(k) = A$ implies $n_i^A(k) < \tau_i \deg_i$ and $x_i(k) = B$ implies $n_i^A(k) \geq \tau_i \deg_i$. Note that this notion of equilibrium is equivalent to a pure strategy Nash equilibrium in the corresponding network game.

8.3 Convergence results

We investigate the equilibrium convergence properties of the agent-based threshold models in (8.2) and (8.3) that we defined in the previous Section.

Before providing the main results, we precisely define the nature of the asynchronous dynamics. We require only that at any given time step, each agent is guaranteed to be active at some finite future time. Let i^k denote the agent who is active at time k and let $(i^k)_{k=0}^{\infty}$ denote a sequence of active agents. We say that such a sequence is *persistent* if for every agent $j \in \mathcal{V}$ and every time $k \geq 0$, there exists some finite later time $k' > k$ at which agent j is again active ($i^{k'} = j$).

Assumption 8.1. *Every activation sequence driving the dynamics in* (8.2) *or* (8.3) *is persistent.*

Remark 8.1. *In stochastic settings, Assumption 8.1 holds almost surely whenever agents activate infinitely many times with probability one, for example, if each agent is activated at a rate determined by a Poisson process.*

We divide the convergence analysis into two main parts corresponding to the cases of anti-coordinating and coordinating agents. In what follows, we use $\mathbf{1}$ to denote the n-dimensional vector containing all ones.

8.3.1 All agents are anti-coordinating

Theorem 8.1. *Every network of anti-coordinating agents who update asynchronously under Assumption 8.1 will reach an equilibrium state in finite time.*

The sketch of the proof is as follows. We begin by showing that an arbitrary network game $\Gamma = (\mathcal{G}, \frac{1}{2}\mathbf{1}, -)$ consisting of anti-coordinating agents in which the threshold of each agent is $\frac{1}{2}$ will reach an equilibrium in finite time. Then we extend the result to a heterogeneous-threshold network game $\Gamma = (\mathcal{G}, \tau, -)$ by constructing a homogeneous-threshold *augmented* network game $\hat{\Gamma} = (\hat{\mathcal{G}}, \frac{1}{2}\mathbf{1}, -)$ that is dynamically equivalent to Γ. We complete the proof by showing that if the augmented network game $\hat{\Gamma}$ reaches an equilibrium, then Γ does as well. The following lemma establishes convergence of the homogeneous-threshold network game $(\mathcal{G}, \frac{1}{2}\mathbf{1}, -)$.

Lemma 8.1. *Every network of anti-coordinating agents who update asynchronously under Assumption 8.1, with $\tau_i = \frac{1}{2}$ for each agent $i \in \mathcal{V}$, will reach an equilibrium state in finite time.*

The proof of the lemma revolves around the following potential function $\Phi(k)$ defined by

$$\Phi(k) = \sum_{i=1}^{n} \Phi_i(k), \qquad \Phi_i(k) = \begin{cases} n_i^A(k) - \hat{n}_i^A & \text{if } x_i(k) = A \\ \hat{n}_i^A + 1 - n_i^A(k) & \text{if } x_i(k) = B \end{cases} \tag{8.4}$$

where $\hat{n}_i^A$ denotes the maximum number of A-neighbors of agent i that will not cause agent i to switch to B when playing A. The proof follows from the fact that the function is lower bounded and decreases every time an agent in the network switches strategies.

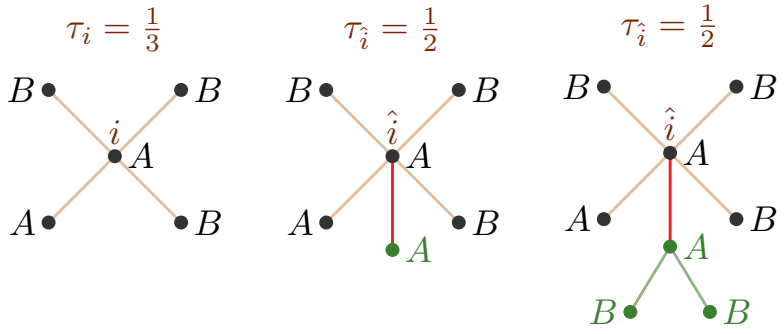


Figure 8.1: Agents i and $\hat{i}$ have thresholds $\tau_i = \frac{1}{3}$ and $\tau_{\hat{i}} = \frac{1}{2}$, but because an extra A-neighbor has been added to $\hat{i}$, they are dynamically equivalent. To ensure that the added node does not switch, two B neighbors are added who also will not switch, forming a *V-block*.

Proof of Lemma 8.1. Consider the function $\Phi(k)$ in (8.4) which is clearly lower bounded by $\Phi(k) \geq -\sum_{i=1}^{n} \deg_i$ for all k. Consider a time step k, and let i denote the index of the active agent at that time. One of the following three cases must happen:

1) Agent i does not switch strategies at time $k+1$. This implies $\Phi(k+1) = \Phi(k)$.

2) Agent i switches from A to B at time $k+1$. This implies $n_i^A(k) \geq \hat{n}_i^A + 1$. Then, since $n_i^A(k) = n_i^A(k+1)$, we have

$$\begin{aligned} \Phi_i(k+1) - \Phi_i(k) &= \hat{n}_i^A + 1 - n_i^A(k) - n_i^A(k) + \hat{n}_i^A \\ &= 2(\hat{n}_i^A - n_i^A(k)) + 1 \leq -1. \end{aligned} \tag{8.5}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\begin{aligned} \Phi_j(k+1) - \Phi_j(k) &= n_j^A(k+1) - \hat{n}_j^A - n_j^A(k) + \hat{n}_j^A \\ &= -1, \end{aligned} \tag{8.6}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = -n_j^A(k+1) + n_j^A(k) = 1. \tag{8.7}$$

According to (8.3), the fact that agent i switches from A to B at time $k+1$ implies $n_i^A(k) \geq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (8.5), (8.6), and (8.7), we have

$$\begin{aligned}
&\Phi(k+1) - \Phi(k) \\
&\quad = \sum_{j \in \mathcal{N}_i \cup \{i\}} \Phi_j(k+1) - \Phi_j(k) \\
&\quad = \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} \underbrace{-n_i^A(k) + (\deg_i - n_i^A(k))}_{\leq 0} \\
&\quad \leq -1.
\end{aligned} \tag{8.8}$$

3) Agent i switches from B to A at time $k+1$. This implies $n_i^A(k) \leq \hat{n}_i^A$. Hence,

$$\Phi_i(k+1) - \Phi_i(k) = 2(n_i^A(k) - \hat{n}_i^A) - 1 \leq -1. \tag{8.9}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = n_j^A(k+1) - n_j^A(k) = 1, \tag{8.10}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = -n_j^A(k+1) + n_j^A(k) = -1. \tag{8.11}$$

According to (8.3), the fact that agent i switches from B to A at time $k+1$ implies $n_i^A(k) \leq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (8.9), (8.10), and (8.11), we have

$$\begin{aligned}
&\Phi(k+1) - \Phi(k) \\
&\quad = \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} + \underbrace{n_i^A(k) - (\deg_i - n_i^A(k))}_{\leq 0} \\
&\quad \leq -1.
\end{aligned}$$

By summarizing the above three cases, we have that

$$\Phi(k+1) \leq \Phi(k) \qquad \forall k \geq 0. \tag{8.12}$$

Moreover, we have shown that every time an agent switches strategies, the function $\Phi(k)$ decreases by at least one. The case where all thresholds are equal to $\frac{1}{2}$ is thus a generalized ordinal potential game, by the definition given in [108]. However, as shown in [8], this does not necessarily imply convergence to an equilibrium. Hence, we complete the proof by contradiction.

Assume that the network does not reach an equilibrium in finite time. Hence, at every time step $k = 0, 1, \ldots$, there exists an agent i^k whose strategy violates its threshold. Denote by $\tilde{k}$ the first time after k at which agent i^k is active. The existence of $\tilde{k}$ is guaranteed by Assumption 8.1. At time $\tilde{k}$, agent i^k's threshold either remains violated, implying the agent will switch strategies at time $\tilde{k}+1$, or is no longer violated, implying that some neighbors have changed their strategies during the time interval $[k+1, \tilde{k}]$. Hence, at least one switch occurs in each interval $\mathcal{I}^k = [k+1, \tilde{k}+1]$. Now consider the sequence of intervals $\mathcal{I}^{k_1}, \mathcal{I}^{k_2}, \ldots$ where the indices $k_j, j = 1, 2, \ldots$, are such that $k_{j+1} > \tilde{k}_j + 1$.

This sequence is infinite, and the intersection of each pair of intervals from the sequence is empty. Therefore, an infinite number of switches occur in the network over time. Namely, there exists an infinite time sequence $(\kappa^j)_{j=1}^{\infty}, \kappa^j \in \mathcal{I}^{k_j}$, such that an agent switches strategies at each κ^j. Hence, either Case 2 or 3 occurs at each κ^j, resulting in $\Phi(\kappa^j + 1) \leq \Phi(\kappa^j) - 1$. Hence, in view of (8.12),

$$\Phi(k) \leq \Phi(\kappa^j) - 1 \qquad \forall k \geq \kappa^j + 1. \tag{8.13}$$

Since (8.13) holds for all $j = 1, 2, \ldots$, we get that

$$\Phi(k) \leq \Phi(\kappa^1) - j \qquad \forall k \geq \kappa^j + 1 \quad \forall j \in \mathbb{N}.$$

The above inequality implies that Φ is not lower bounded, which is a contradiction. Hence, the proof is complete. □

To motivate our approach for extending this result to an arbitrary distribution of thresholds, consider for example an agent i with 4 neighbors whose threshold is $\frac{1}{3}$. When playing A, this agent can tolerate up to 1 A-neighbor (Figure 8.1-a), but 2 or more will cause a switch to B. Similarly, when playing B, the agent needs at least 2 A-neighbors to remain playing B, while 1 or fewer will cause a switch to A. Now consider an agent $\hat{i}$ whose threshold is $\frac{1}{2}$ but who has one additional neighbor who always plays A, for a total of 5 neighbors, as shown in Figure 8.1-b. When playing A, this agent can tolerate up to 2 A-neighbors before switching to B, and as a B-agent needs at least 3 A-neighbors, while 2 or fewer will cause a switch to A. Notice however, that with respect to the original 4 neighbors, the dynamics of agents i and $\hat{i}$ are indistinguishable. It turns out that whenever $\tau_i < \frac{1}{2}$, by adding a sufficient number of fixed A-neighbors, we can always construct a dual node $\hat{i}$ with threshold $\frac{1}{2}$ whose dynamics are equivalent to the dynamics of the original node i. Moreover, if $\tau_i > \frac{1}{2}$ we can achieve the same result by adding fixed B-neighbors. To ensure that the added nodes do not change strategies, we simply add two opposite strategy neighbors to these nodes (Figure 8.1-c). It is then straightforward to show that the strategies of all added nodes remain constant. We now formalize this argument for arbitrary networks of anti-coordinating agents, using some techniques similar to those that have already proven useful in studying the convergence of synchronous networks [2].

We define the *augmented network game* $\hat{\Gamma} := (\hat{\mathcal{G}}, \frac{1}{2}\mathbf{1}, -)$ based on Γ as follows. Let $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$. Define a *$V$-block* as a triplet of nodes $\{v_1, v_2, v_3\} \subseteq \hat{\mathcal{V}}$ along with the edges $\{\{v_1, v_2\}, \{v_1, v_3\}\} \subseteq \hat{\mathcal{E}}$. For each agent $i \in \mathcal{V}$, we introduce a *dual agent* $\hat{i} \in \hat{\mathcal{V}}$ with the same initial strategy, i.e., $x_{\hat{i}}(0) = x_i(0)$, and with $z_{\hat{i}} = z_i$. Corresponding to each dual agent $\hat{i}$, there are m_i number of V-blocks in $\hat{\mathcal{G}}$ such that the v_1-node of each block is connected to $\hat{i}$, with m_i being defined as follows: if $\tau_i = \frac{1}{2}$, then $m_i = 0$; otherwise, m_i depends on which one of the following three conditions on τ_i holds:

$$m_i = \begin{cases} |(1 - 2\tau_i)\deg_i| & \tau_i \deg_i \in \mathbb{Z} \\ |\deg_i - r - 1| & \exists r \in 2\mathbb{Z} : \frac{r}{2} < \tau_i \deg_i < \frac{r+1}{2} \\ |\deg_i - r| & \exists r \in 2\mathbb{Z}+1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2} \end{cases} \tag{8.14}$$

where $2\mathbb{Z}$ and $2\mathbb{Z}+1$ denote the set of even and odd numbers, respectively. If $\tau_i < \frac{1}{2}$, then the initial strategies of each V-block connected to the dual agent $\hat{i}$ are $x_{v_1}(0) = A$ and $x_{v_2}(0) = x_{v_3}(0) = B$; and if $\tau_i > \frac{1}{2}$, then $x_{v_1}(0) = B$ and $x_{v_2}(0) = x_{v_3}(0) = A$. In total, $\hat{\mathcal{V}}$ has $n + \sum_{i=1}^{n} m_i$ agents, the

thresholds of all of which are set to $\frac{1}{2}$. For $\hat{\mathcal{E}}$, in addition to the edges for the V-blocks, there is an edge between any two dual agents $\hat{i}$ and $\hat{j}$ in $\hat{\mathcal{V}}$, if and only if there is an edge between i and j in $\mathcal{V}$.

Lemma 8.2. *The strategy of each V-block agent is fixed over time.*

Proof. We prove by contradiction. Assume there exists some time when the strategy of one of the V-block agents changes. Let r denote the first time this happens, and denote the V-block agent who changes her strategy by i. If i is a v_1-node and $x_i(0) = A$, then $x_i(r-1) = A$ and $x_i(r) = B$. Agent i has two neighbors in the V-block who each play strategy B at time $r-1$. Since $\deg_i = 3$, we have

$$\left.\begin{aligned} n_i^A(r-1) &\leq 1 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = A,$$

which is a contradiction. If $x_i(0) = B$, then agent i has two A-neighbors in the V-block and $x_i(r) = A$. It follows that

$$\left.\begin{aligned} n_i^A(r-1) &\geq 2 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = B,$$

again a contradiction. Now if i is either a v_2 or a v_3 node, and $x_i(0) = A$, then $x_i(r-1) = A$, and i has only one neighbor, v_1, whose strategy is B at time $r-1$. Hence,

$$\left.\begin{aligned} n_i^A(r-1) &= 0 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = A,$$

a contradiction. If on the other hand, $x_i(0) = B$, then

$$\left.\begin{aligned} n_i^A(r-1) &= 1 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = B,$$

which is a contradiction and completes the proof. □

The following lemma takes the first step towards establishing equivalence of the dynamics between the original and augmented network games. In particular, we need to show that the thresholds of $\frac{1}{2}$ in the augmented network will be satisfied (respectively violated) whenever the thresholds of the corresponding agents in the original network are satisfied (respectively violated).

Lemma 8.3. *Let n_i^A denote an instance of $n_i^A(k)$ for some agent i. If $\tau_i < \frac{1}{2}$, then*

$$\operatorname{sign}(n_i^A - \tau_i \deg_i) = \operatorname{sign}\left(n_i^A + m_i - \frac{1}{2}(\deg_i + m_i)\right), \tag{8.15}$$

and if $\tau_i > \frac{1}{2}$, then

$$\operatorname{sign}(n_i^A - \tau_i \deg_i) = \operatorname{sign}\left(n_i^A - \frac{1}{2}(\deg_i + m_i)\right). \tag{8.16}$$

Proof. First consider the situation when $\tau_i < \frac{1}{2}$. In general, one of the following cases takes place:

1) $\tau_i \deg_i \in \mathbb{Z}$*:* If $n_i^A = \tau_i \deg_i$, using (8.14) we have

$$n_i^A = \frac{1}{2}(\deg_i - m_i) \Rightarrow n_i^A + m_i = \frac{1}{2}(\deg_i + m_i).$$

The cases $n_i^A < \tau_i \deg_i$ and $n_i^A > \tau_i \deg_i$ can be shown using the same approach, which verifies (8.15) for this case.

2) $\exists r \in 2\mathbb{Z} : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$*:* Here, $\tau_i \deg_i \notin \mathbb{Z}$ implies that $n_i^A \neq \tau_i \deg_i$, so we need only to check the inequality cases. Using (8.14), $n_i^A > \tau_i \deg_i$ implies

$$\left.\begin{array}{l} n_i^A > \dfrac{r}{2} \\ \dfrac{r}{2}, n_i^A \in \mathbb{Z} \end{array}\right\} \Rightarrow n_i^A \geq \frac{r}{2} + 1 = \frac{1}{2}(\deg_i - m_i) + \frac{1}{2} \Rightarrow n_i^A + m_i > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A < \frac{r+1}{2} = \frac{1}{2}(\deg_i - m_i) \Leftrightarrow n_i^A + m_i < \frac{1}{2}(\deg_i + m_i).$$

Hence, (8.15) is confirmed for this case.

3) $\exists r \in 2\mathbb{Z}+1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$*:* Again, $\tau_i \deg_i \notin \mathbb{Z}$ implies that $n_i^A \neq \tau_i \deg_i$. Then $n_i^A > \tau_i \deg_i$ implies

$$n_i^A > \frac{r}{2} = \frac{1}{2}(\deg_i - m_i) \Leftrightarrow n_i^A + m_i > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$\left.\begin{array}{l} n_i^A < \dfrac{r+1}{2} \\ n_i^A \in \mathbb{Z}, r \in 2\mathbb{Z}+1 \end{array}\right\} \Rightarrow n_i^A \leq \frac{r-1}{2} \Leftrightarrow n_i^A \leq \frac{1}{2}(\deg_i - m_i) - \frac{1}{2}$$
$$\Rightarrow n_i^A + m_i < \frac{1}{2}(\deg_i + m_i).$$

Hence, (8.15) holds for this case and for all $\tau_i < \frac{1}{2}$.

If $\tau_i > \frac{1}{2}$, then one of the following occurs:

1) $\tau_i \deg_i \in \mathbb{Z}$*:* If $n_i^A = \tau_i \deg_i$, then (8.14) implies $n_i^A = \frac{1}{2}(\deg_i + m_i)$. The cases $n_i^A < \tau_i \deg_i$ and $n_i^A > \tau_i \deg_i$ can be shown using the same approach, which verifies (8.16) for this case.

2) $\exists r \in 2\mathbb{Z} : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$*:* First, we know that $n_i^A \neq \tau_i \deg_i$. Using (8.14), $n_i^A > \tau_i \deg_i$ implies

$$n_i^A \geq \frac{r}{2} + 1 = \frac{1}{2}(\deg_i + m_i) + \frac{1}{2} > \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A < \frac{r+1}{2} = \frac{1}{2}(\deg_i + m_i) \Leftrightarrow n_i^A < \frac{1}{2}(\deg_i + m_i).$$

Hence, (8.16) holds for this case.

3) $\exists r \in 2\mathbb{Z}+1 : \frac{r}{2} \leq \tau_i \deg_i < \frac{r+1}{2}$*:* Once again, we know that $n_i^A \neq \tau_i \deg_i$. Using (8.14), $n_i^A > \tau_i \deg_i$ implies

$$n_i^A > \frac{r}{2} = \frac{1}{2}(\deg_i + m_i),$$

and $n_i^A < \tau_i \deg_i$ implies

$$n_i^A \leq \frac{r-1}{2} = \frac{1}{2}(\deg_i + m_i) - \frac{1}{2} < \frac{1}{2}(\deg_i + m_i).$$

Hence, (8.16) holds for this case and for all $\tau_i > \frac{1}{2}$, which completes the proof. □

Next, we show in Lemma 8.4 that if whenever an agent in $\mathcal{G}$ is activated, its dual in $\hat{\mathcal{G}}$ also is activated (while neglecting the time steps that a V-block agent is active), then the dynamics of each node in $\mathcal{G}$ are the same as the dynamics of its dual node in $\hat{\mathcal{G}}$ (again while neglecting the time steps that a V-block agent is active).

Consider the network $\mathcal{G}$ and let $i_{\mathcal{G}}^k$ denote the active agent at time k. Correspondingly, denote by $(i_{\mathcal{G}}^k)_{k=0}^{\infty}$, the sequence of active agents in $\mathcal{G}$. Similarly define $(i_{\hat{\mathcal{G}}}^k)_{k=0}^{\infty}$ as the sequence of active agents in $\hat{\mathcal{G}}$. Consider $(i_{\hat{\mathcal{G}}}^k)_{k=0}^{\infty}$ and exclude those agents $i_{\hat{\mathcal{G}}}^k$ that belong to one of the V-blocks, to get the subsequence $(i_{\hat{\mathcal{G}}}^{h_k})_{k=0}^{\infty}$. Denote the sequence of superscripts of $(i_{\hat{\mathcal{G}}}^{h_k})_{k=0}^{\infty}$ by $(h_k)_{k=0}^{\infty}$ which corresponds to the times at which the non-V-block agents in $\hat{\mathcal{G}}$ are active.

Lemma 8.4. *If* $(i_{\mathcal{G}}^k)_{k=0}^{\infty} = (i_{\hat{\mathcal{G}}}^{h_k})_{k=0}^{\infty}$*, then for* $k = 0, 1, \ldots$*, it holds that*

$$x_i(k) = x_{\hat{i}}(h_k) \qquad \forall i \in \mathcal{V}. \tag{8.17}$$

where $\hat{i} \in \hat{\mathcal{V}}$ *is the dual of agent* i*.*

Proof. The proof is done via induction on k. By the definition of $\hat{\mathcal{G}}$, (8.17) holds for $k = 0$. Assume that (8.17) holds for $k = r \in \mathbb{Z}_{\geq 0}$.

Consider agent $i_{\mathcal{G}}^r$ and its dual $\hat{i}_{\mathcal{G}}^r$ whose threshold and degree are $\frac{1}{2}$ and $\deg_i + m_i$, respectively. Agent $i_{\mathcal{G}}^r$ updates at time $k = r+1$, and agent $\hat{i}_{\mathcal{G}}^r$ updates at $k_{\hat{\mathcal{G}}} = h_r + 1$ where $k_{\hat{\mathcal{G}}}$ denotes the time in the augmented network game $\hat{\Gamma}$. If $\tau_i = \frac{1}{2}$, then both agents have the same threshold and number of A-neighbors. Hence, they update to the same strategy at the next time step. If $\tau_i < \frac{1}{2}$, then in view of Lemma 8.2 and since (8.17) holds for $k = r$, $\hat{i}_{\mathcal{G}}^r$ has $n_{i^r}^A(r) + m_i$ A-neighbors. Therefore, according to (8.15) in Lemma 8.3, $\hat{i}_{\mathcal{G}}^r$ updates to the same strategy that $i_{\mathcal{G}}^r$ does. On the other hand, if $\tau_i > \frac{1}{2}$, then $\hat{i}_{\mathcal{G}}^r$ has $n_{i^r}^A(r)$ A-neighbors. Hence, according to (8.16) in Lemma 8.3, $\hat{i}_{\mathcal{G}}^r$ updates to the same strategy that $i_{\mathcal{G}}^r$ does. Therefore, in all cases, agent $\hat{i}_{\mathcal{G}}^r$ updates to the same strategy that agent $i_{\mathcal{G}}^r$ does. That is,

$$x_{i_{\mathcal{G}}^r}(r+1) = x_{\hat{i}_{\mathcal{G}}^r}(h_r + 1). \tag{8.18}$$

On the other hand, since no other agent has become active at times h_r or r,

$$x_i(r+1) = x_i(r) \qquad \forall i \in \mathcal{V} - \{i_{\mathcal{G}}^r\}, \tag{8.19}$$

$$x_{\hat{i}}(h_r+1) = x_{\hat{i}}(h_r) \qquad \forall i \in \mathcal{V} - \{i^r_{\mathcal{G}}\}. \tag{8.20}$$

Due to the induction statement for $k = r$, it holds that $x_i(r) = x_{\hat{i}}(h_r)$ for all $i \in \mathcal{V} - \{i^r_{\mathcal{G}}\}$. Hence, (8.19) and (8.20) result in

$$x_i(r+1) = x_{\hat{i}}(h_r+1) \qquad \forall i \in \mathcal{V} - \{i^r_{\mathcal{G}}\}.$$

Therefore, according to (8.18),

$$x_i(r+1) = x_{\hat{i}}(h_r+1) \qquad \forall i \in \mathcal{V}. \tag{8.21}$$

Now since at each of the time steps $h_r+1, h_r+2, \ldots, h_{r+1}-1$, the active agent is a V-block agent whose strategy remains fixed by Lemma 8.2, (8.21) results in

$$x_i(r+1) = x_{\hat{i}}(h_{r+1}) \qquad \forall i \in \mathcal{V}.$$

Hence, (8.17) holds for $k = r+1$, which completes the proof by induction. □

The remaining step in proving Theorem 8.1 is to show that agents with arbitrary thresholds will indeed reach an equilibrium state in finite time.

Proof of Theorem 8.1. Towards a proof by contradiction, suppose that the original network game never converges, i.e., there exists an agent $j \in \mathcal{V}$ such that

$$\forall k^*, \big(\exists k > k^* : x_j(k) \neq x_j(k^*)\big).$$

Construct the sequence of active agents $(i^k_{\hat{\mathcal{G}}})_{k=0}^{\infty}$ by inserting an agent $\hat{i}$ uniformly at random from the set of augmented nodes $\hat{\mathcal{V}} - \mathcal{V}$ after every n elements of the sequence $(i^k_{\mathcal{G}})_{k=0}^{\infty}$. This is clearly a persistent activation sequence on the network $\hat{\mathcal{G}}$. By Lemma 8.1, we know that

$$\exists \tilde{k}^* : \big(\forall k > \tilde{k}^*, x_{\hat{j}}(k) = x_{\hat{j}}(\tilde{k}^*)\big). \tag{8.22}$$

On the other hand, by eliminating the V-block agents in $(i^k_{\hat{\mathcal{G}}})_{k=0}^{\infty}$, we arrive at $(i^k_{\mathcal{G}})_{k=0}^{\infty}$. Hence, in view of Lemma 8.4, (8.22) implies that

$$\exists k^* : \big(\forall k > k^*, x_j(k) = x_j(k^*)\big),$$

which contradicts our initial statement. Therefore, $x(k)$ will reach an equilibrium in finite time. □

As a result of Theorem 8.1, we have the following corollary, which to the best of our knowledge has not been shown in the literature to date.

Corollary 8.1. *Every network of anti-coordinating agents admits a pure strategy Nash equilibrium.*

8.3.2 All agents are coordinating

Theorem 8.2. *Every network of coordinating agents who update asynchronously under Assumption 8.1 will reach an equilibrium state in finite time.*

The proof of Theorem 8.2 follows similar steps as the anti-coordinating case. The key difference is that the potential function becomes

$$\Phi(k) = \sum_{i=1}^{n} \Phi_i(k), \qquad \Phi_i(k) = \begin{cases} \check{n}_i^A - n_i^A(k) & \text{if } x_i(k) = A \\ n_i^A(k) - \check{n}_i^A + 1 & \text{if } x_i(k) = B \end{cases}, \tag{8.23}$$

where $\check{n}_i^A$ is defined as the minimum number of A-neighbors required for an A-playing agent to continue playing A. The maximum number of A-neighbors that a B agent can tolerate before switching to A is then given by $\check{n}_i^A - 1$. As shown in the following lemma, this function also decreases by at least 1 with every change of strategy for the network game $\Gamma := (\mathcal{G}, \frac{1}{2}\mathbf{1}, +)$.

Lemma 8.5. *Every network of coordinating agents who update asynchronously under Assumption 8.1, with $\tau_i = \frac{1}{2}$ for each agent $i \in \mathcal{V}$, will reach an equilibrium state in finite time.*

Proof. Consider the function $\Phi(k) = \sum_{i=1}^{n} \Phi_i(k)$, where Φ_i is defined in (8.23). Clearly $\Phi(k)$ is lower bounded by $\Phi(k) \geq -\sum_{i=1}^{n} \deg_i$ for all k. Consider a time step k, and let i denote the active agent at that time. One of the following three cases must happen:

1) Agent i does not switch strategies at time $k+1$. This implies $\Phi(k+1) = \Phi(k)$.

2) Agent i switches from A to B at time $k+1$. This implies $n_i^A(k) \leq \check{n}_i^A - 1$. Hence, since $n_i^A(k) = n_i^A(k+1)$, we have

$$\begin{aligned} \Phi_i(k+1) - \Phi_i(k) &= n_i^A(k) - \check{n}_i^A + 1 - \check{n}_i^A + n_i^A(k) \\ &= 2(n_i^A(k) - \check{n}_i^A) + 1 \leq -1. \end{aligned} \tag{8.24}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\begin{aligned} \Phi_j(k+1) - \Phi_j(k) &= \check{n}_i^A - n_i^A(k+1) - \check{n}_i^A + n_i^A(k) \\ &= 1, \end{aligned} \tag{8.25}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = n_i^A(k+1) - n_i^A(k) = -1. \tag{8.26}$$

According to (8.2), the fact that agent i switches from A to B at time $k+1$ implies $n_i^A(k) \leq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (8.24), (8.25), and (8.26), we have

$$\begin{aligned} &\Phi(k+1) - \Phi(k) \\ &\quad = \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} + \underbrace{n_i^A(k) - (\deg_i - n_i^A(k))}_{\leq 0} \\ &\quad \leq -1. \end{aligned} \tag{8.27}$$

3) Agent i switches from B to A at time $k+1$. This implies $n_i^A(k) \geq \check{n}_i^A$. Hence,

$$\Phi_i(k+1) - \Phi_i(k) = 2(\check{n}_i^A - n_i^A(k)) - 1 \leq -1. \tag{8.28}$$

Moreover, for each $j \in \mathcal{N}_i$, if $x_j(k) = A$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = -n_j^A(k+1) + n_j^A(k) = -1, \tag{8.29}$$

and if $x_j(k) = B$, it holds that

$$\Phi_j(k+1) - \Phi_j(k) = n_j^A(k+1) - n_j^A(k) = 1. \tag{8.30}$$

According to (8.3), the fact that agent i switches from B to A at time $k+1$ implies $n_i^A(k) \geq \frac{1}{2}\deg_i$, regardless of how z_i is defined. Hence, by combining (8.28), (8.29), and (8.30), we have

$$\begin{aligned} &\Phi(k+1) - \Phi(k) && (8.31)\\ &= \underbrace{\Phi_i(k+1) - \Phi_i(k)}_{\leq -1} \underbrace{-n_i^A(k) + (\deg_i - n_i^A(k))}_{\leq 0} && (8.32)\\ &\leq -1. && (8.33) \end{aligned}$$

By summarizing the above three cases, we have that

$$\Phi(k+1) \leq \Phi(k) \qquad \forall k \geq 0. \tag{8.34}$$

Moreover, we have shown that every time an agent switches strategies, the function $\Phi(k)$ decreases by at least one. The rest of the proof follows in the same way as that of Lemma 8.1. □

By following the same process of constructing the network augmentation for anti-coordinating agents, we are able to extend the result of Lemma 8.5 to a network game with arbitrary thresholds. We define the augmented (coordinating) network game $\hat{\Gamma} := (\hat{\mathcal{G}}, \frac{1}{2}\mathbf{1}, +)$ based on the (coordinating) network game Γ, in the same way we defined the augmented network game for anti-coordinating agents, but with the following difference: If $\tau_i < \frac{1}{2}$, then the initial strategies of each V-block connected to the dual agent $\hat{i}$ are $x_{v_1}(0) = x_{v_2}(0) = x_{v_3}(0) = A$, and if $\tau_i > \frac{1}{2}$, then $x_{v_1}(0) = x_{v_2}(0) = x_{v_3}(0) = B$. Similar to Lemma 8.2, the following lemma guarantees the invariance of the strategies of the V-block agents.

Lemma 8.6. *The strategy of each (coordinating) V-block agent is fixed over time.*

Proof. The proof is done via contradiction. Assume there exists some time when the strategy of one of the V-block agents changes. Let r denote the first time this happens, and denote the V-block agent who changes her strategy by i. If i is a v_1-node and $x_i(0) = A$, then $x_i(r-1) = A$ and $x_i(r) = B$. Agent i has two neighbors in the V-block who each play strategy A at time $r-1$. Since $\deg_i = 3$, we have

$$\left.\begin{aligned} n_i^A(r-1) &\geq 2 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = A,$$

which is a contradiction. If $x_i(0) = B$, then agent i has two B-neighbors in the V-block and $x_i(r) = A$. It follows that

$$\left.\begin{aligned} n_i^A(r-1) &\leq 1 \\ \tau_i \deg_i &= \frac{3}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = B,$$

again a contradiction. Now if i is either a v_2 or a v_3 node, and $x_i(0) = A$, then $x_i(r-1) = A$, and i has only one neighbor, v_1, whose strategy is A at time $r-1$. Hence,

$$\left.\begin{aligned} n_i^A(r-1) &= 1 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) > \tau_i \deg_i \Rightarrow x_i(r) = A,$$

a contradiction. If on the other hand, $x_i(0) = B$, then

$$\left.\begin{aligned} n_i^A(r-1) &= 0 \\ \tau_i \deg_i &= \frac{1}{2} \end{aligned}\right\} \Rightarrow n_i^A(r-1) < \tau_i \deg_i \Rightarrow x_i(r) = B,$$

which is a contradiction and completes the proof. □

Next, since Lemma 8.3 is independent of the type of agents, i.e., coordinating or anti-coordinating, it can be used here as well. Moreover, because of Lemma 8.6, the result of Lemma 8.4 can be readily extended to a network of coordinating agents. With these lemmas in hand, and with the help of Lemma 8.5, the proof of Theorem 8.2 can be done in the same way as that of Theorem 8.1.

Theorem 8.2 also leads to a corollary on existence of equilibria for networks of coordinating agents, but in this case we confirm an already known result, e.g., [8].

Corollary 8.2. *Every network of coordinating agents admits a pure strategy Nash equilibrium.*

The following corollary follows directly from Theorems 8.1 and 8.2, and the fact that a network of homogeneous agents can always be described as all coordinating or all anti-coordinating.

Corollary 8.3. *Every network of homogenous agents who update asynchronously under Assumption 8.1 will reach an equilibrium state in finite time.*

8.3.3 Coordinating and anti-coordinating agents coexist

Given the previous two results, the natural question arises of what we can say about convergence when both coordinating and anti-coordinating agents are present in a network. Although there may be particular configurations that converge, we can demonstrate that this will not hold in general with a simple example on a network consisting of only two agents connected to each other, namely a path of length one, i.e., $\mathcal{V} := \{1, 2\}, \mathcal{E} := \{(1, 2), (2, 1)\}$. Suppose that agent 1 is coordinating, agent 2 is anti-coordinating, and their thresholds are both equal to $\frac{1}{2}$. Given their respective update rules in (8.2) and (8.3), agent 1 will always switch to match agent 2 while agent 2 will always switch to oppose the strategy of agent 1. In other words, the following transitions will occur with probability one: $(A, A) \to (A, B) \to (B, B) \to (B, A) \to (A, A)$, resulting in a never-ending cycle. This formulation is equivalent to a repeated version of the well-known matching pennies game. From this we can conclude that convergence is not a trivial result to be expected in all cases, and can only be guaranteed when all agents are either coordinating or anti-coordinating.

8.3.4 Convergence time

The following corollary follows from the fact that the potential function $\Phi(k)$ is bounded from above and below and decreases by at least one every time an agent switches strategies.

Corollary 8.4. *Every network of all coordinating or all anti-coordinating agents will reach an equilibrium state after no more than* $6|\mathcal{E}|$ *agent switches.*

Proof. To compute the maximum number of times any agent switches strategies before such a network reaches an equilibrium, we consider the augmented network game $\hat{\Gamma}$, which will undergo the same sequence of agent switches as the original network game Γ, provided that the dual agents in $\hat{\mathcal{V}}$ activate in the same order as the corresponding agents in $\mathcal{V}$. From (8.8) in the proof of Lemma 8.1 and (8.27) in the proof of Lemma 8.5, we know that whenever an agent $i \in \hat{\mathcal{V}}$ switches strategies, $\Phi(k+1) - \Phi(k) \leq -1$. Otherwise, $\Phi(k)$ remains constant. It follows that the total number of agent switches in $\hat{\Gamma}$ is bounded from above by $\Phi(0) - \Phi(k^*)$, where k^* is the time at which the network reaches an equilibrium. To obtain such a bound, we start by decomposing the augmented network into three disjoint sets of agents such that $\hat{\mathcal{V}} = \hat{\mathcal{V}}_0 \cup \hat{\mathcal{V}}_1 \cup \hat{\mathcal{V}}_{23}$, where $\hat{\mathcal{V}}_0$ denotes the dual agents corresponding to the oringinal agents $\mathcal{V}$, $\hat{\mathcal{V}}_1$ denotes the set of v_1 agents in the V-blocks, and $\hat{\mathcal{V}}_{23}$ denotes the set of v_2 and v_3 agents in the V-blocks (we refer the reader to the proof of Theorem 8.1 for definitions of the augmented network). We can now expand the expression for the upper bound as follows:

$$\Phi(0) - \Phi(k^*) = \sum_{i \in \hat{\mathcal{V}}} \Phi_i(0) - \Phi_i(k^*) = \sum_{i \in \hat{\mathcal{V}}_0} \Phi_i(0) - \Phi_i(k^*) + \sum_{i \in \hat{\mathcal{V}}_1} \Phi_i(0) - \Phi_i(k^*) + \sum_{i \in \hat{\mathcal{V}}_{23}} \Phi_i(0) - \Phi_i(k^*). \tag{8.35}$$

Since the V-block agents never change strategies (by Lemmas 8.2 and 8.6), $\Phi_i(k)$ is constant for all agents in $\hat{\mathcal{V}}_{23}$. The final term in (8.35) is therefore equal to zero. The agents in $\hat{\mathcal{V}}_1$ each have one neighbor in $\hat{\mathcal{V}}_0$ who might change strategies (the other two neighbors are in $\hat{\mathcal{V}}_{23}$ and remain fixed). Since $n_i^A(k)$ can change by at most one for these agents, it follows that the maximum change in $\Phi_i(k)$ for such an agent is one. Therefore, we have

$$\sum_{i \in \hat{\mathcal{V}}_1} \Phi_i(0) - \Phi_i(k^*) \leq \sum_{i \in \hat{\mathcal{V}}_1} 1 = |\hat{\mathcal{V}}_1| = \sum_{i \in \mathcal{V}} m_i < \sum_{i \in \mathcal{V}} \deg_i, \tag{8.36}$$

since the size of the set $\hat{\mathcal{V}}_1$ is simply the total number of V-blocks (m_i for each agent), and $m_i < \deg_i$ due to (8.14). Next, we consider the set $\hat{\mathcal{V}}_0$ of dual agents. For a network of anti-coordinating agents at time zero, we have for each $\hat{i} \in \hat{\mathcal{V}}_0$

$$\Phi_{\hat{i}}(0) = \begin{cases} n_{\hat{i}}^A(0) - \hat{n}_{\hat{i}}^A \leq \deg_{\hat{i}} - \frac{1}{2}\deg_{\hat{i}} + 1 & \text{if } x_{\hat{i}}(0) = A \\ \hat{n}_{\hat{i}}^A + 1 - n_{\hat{i}}^A(0) \leq \frac{1}{2}\deg_{\hat{i}} + 1 - 0 & \text{if } x_{\hat{i}}(0) = B \end{cases},$$

where we used the facts that $\tau_{\hat{i}} \deg_{\hat{i}} - 1 \leq \hat{n}_{\hat{i}}^A \leq \tau_{\hat{i}} \deg_{\hat{i}}$ and that the thresholds in the augmented network $\tau_{\hat{i}}$ are all equal to $\frac{1}{2}$. Similarly, for a network of coordinating agents, we have

$$\Phi_{\hat{i}}(0) = \begin{cases} \check{n}_{\hat{i}}^A - n_{\hat{i}}^A(0) \leq \frac{1}{2}\deg_{\hat{i}} + 1 - 0 & \text{if } x_{\hat{i}}(0) = A \\ n_{\hat{i}}^A(0) - \check{n}_{\hat{i}}^A + 1 \leq \deg_{\hat{i}} - \frac{1}{2}\deg_{\hat{i}} + 1 & \text{if } x_{\hat{i}}(0) = B \end{cases},$$

since it holds that $\tau_{\hat{i}} \deg_{\hat{i}} \leq \check{n}_{\hat{i}}^A \leq \tau_{\hat{i}} \deg_{\hat{i}} + 1$. The result is the following upper bound:

$$\Phi_{\hat{i}}(0) \leq \frac{1}{2} \deg_{\hat{i}} + 1 \text{ for all } \hat{i} \in \hat{\mathcal{V}}_0. \tag{8.37}$$

For a network of anti-coordinating agents at equilibrium (at time k^*), we have

$$\Phi_{\hat{i}}(k^*) = \begin{cases} n_{\hat{i}}^A(k^*) - \hat{n}_{\hat{i}}^A \geq 0 - \frac{1}{2} \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = A \\ \hat{n}_{\hat{i}}^A + 1 - n_{\hat{i}}^A(k^*) \geq \frac{1}{2} \deg_{\hat{i}} - \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = B \end{cases}.$$

Similarly, for a network of coordinating agents, we have

$$\Phi_{\hat{i}}(k^*) = \begin{cases} \check{n}_{\hat{i}}^A - n_{\hat{i}}^A(k^*) \geq \frac{1}{2} \deg_{\hat{i}} - \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = A \\ n_{\hat{i}}^A(k^*) - \check{n}_{\hat{i}}^A + 1 \geq 0 - \frac{1}{2} \deg_{\hat{i}} & \text{if } x_{\hat{i}}(k^*) = B \end{cases}.$$

This yields the following lower bound:

$$\Phi_{\hat{i}}(k^*) \geq -\frac{1}{2} \deg_{\hat{i}} \text{ for all } \hat{i} \in \hat{\mathcal{V}}_0. \tag{8.38}$$

Using (8.37) and (8.38), we can bound the change in potential for the dual agents as follows:

$$\sum_{\hat{i} \in \hat{\mathcal{V}}_0} \Phi_{\hat{i}}(0) - \Phi_{\hat{i}}(k^*) \leq \sum_{\hat{i} \in \hat{\mathcal{V}}_0} (\deg_{\hat{i}} + 1).$$

For each dual agent $\hat{i} \in \hat{\mathcal{V}}_0$, let i denote the corresponding original agent in $\mathcal{V}$. Since $\deg_{\hat{i}} = \deg_i + m_i$ and $m_i < \deg_i$ due to (8.14), it holds that $\deg_{\hat{i}} \leq 2 \deg_i - 1$. It follows that

$$\sum_{\hat{i} \in \hat{\mathcal{V}}_0} \Phi_{\hat{i}}(0) - \Phi_{\hat{i}}(k^*) \leq 2 \sum_{i \in \mathcal{V}} \deg_i . \tag{8.39}$$

Substituting (8.36) and (8.39) into (8.35) results in

$$\Phi(0) - \Phi(k^*) \leq 3 \sum_{i \in \mathcal{V}} \deg_i = 6|\mathcal{E}|.$$

Finally, Lemma 8.4 implies that the sequence of agent switches between an original and augmented network are equivalent, as long as the dual agents activate in the same sequence as the agents in the original network. This completes the proof. □

This implies that agents cannot switch an arbitrary number of times before the network reaches equilibrium. It follows that when agent activation times are independent and identically distributed, the upper bound on the expected time to reach equilibrium is linear in the number of edges in the network.

8.4 Synchronous and partially synchronous updating

So far, any network of all coordinating or all anti-coordinating agents is shown to reach an equilibrium state, as long as the agents update asynchronously. However, the importance of asynchronous updating to the convergence results remains an open problem. In this section we show that while full synchrony may not always result in convergence, the results indeed still hold for partial synchrony, in which a random number of agents update at each time step.

8.4.1 Synchronous updating

We show that networks in which updates are fully synchronous may never reach an equilibrium state, by presenting counterexamples with only two agents.

First, suppose that both agents are anti-coordinating and start from the strategy vector (A, A). The agents update synchronously, that is at each time step, both agents activate and update their strategies according to the update rule in (8.3). Therefore, the dynamics will be deterministic, and the following transitions will occur on the strategies of the agents: $(A, A) \to (B, B) \to (A, A)$, resulting in a cycle of length 2.

Now suppose that both agents are coordinating and they start from the strategy vector (A, B). Following the update rule in (8.2), the following transitions take place under the synchronous updating: $(A, B) \to (B, A) \to (A, B)$, again resulting in a cycle of length 2.

The above examples prove that equilibrium convergence is no longer guaranteed if the agents update in full synchrony. However, it is known that any network game governed by synchronous best-response dynamics will reach a cycle of length at most 2, even when both coordinating and anti-coordinating agents coexist in the network [3].

8.4.2 Partially synchronous updating

In order to understand what happens in the case of partially synchronous updates, we need to relax the assumption that only one agent can update at a given time. We must therefore decouple the activation sequence from the discrete-time dynamics, and consider the activations to occur in continuous time. Let $\{t^j\}_{j=1}^{\infty}$ denote a sequence of times and $\{i^j\}_{j=1}^{\infty}$ a sequence of agents such that agent i^j is activated at time t^j. Figure 8.2 shows the relationship of the discrete time steps with the agent activation sequence.

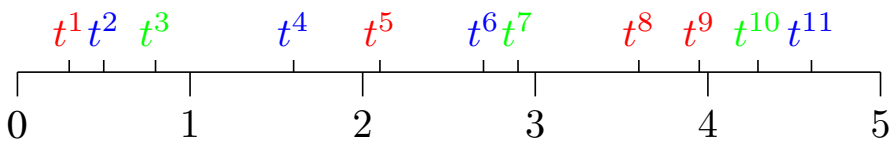


Figure 8.2: In the partially synchronous update model, multiple agents can activate in a single discrete time step. All agents who are active in the absolute time interval $[k, k+1)$ update at time $k+1$ based on the information at time k. The colors red, blue, and green represent three different agents.

It is now possible that multiple agents update between consecutive time steps. In particular, all agents who are active in the time interval $[k, k+1)$ update at time $k+1$ based on the state of the network at time k. Let $\mathcal{A}_k$ denote the set of all agents who are active during the time interval $[k, k+1)$. We can express the partially synchronous dynamics as follows:

$$x_i(k+1) = \begin{cases} x_i^{br}(k) & \text{if } i \in \mathcal{A}_k \\ x_i(k) & \text{otherwise} \end{cases},$$

where $x_i^{br}(k)$ denotes the best response of agent i to the strategies of its neighbors at time k. To provide a general framework for independent stochastic activation sequences, we make the following assumption.

Assumption 8.2. *The inter-activation times for each agent are drawn from mutually independent probability distributions with support on* $\mathbb{R}_{\geq 0}$.

One standard model that satisfies Assumption 8.2 is to use exponential distributions with mean $\frac{1}{\lambda}$. This represents the case in which each agent updates according to a Poisson clock with rate λ, and the expected number of agents updating in one unit of time is λn.

From the previous sections we know that the best response dynamics do not necessarily converge to an equilibrium when the updating is synchronous, yet do converge when the updating is asynchronous. Therefore, the natural question arising is: *how much asynchrony do the partially synchronous best-response dynamics need for convergence?* It turns out that even under the relatively mild assumption on the partially synchronous updates in Assumption 8.2, we still have convergence to an equilibrium state almost surely.

Theorem 8.3. *Every network of all coordinating or all anti-coordinating agents who update with partially synchronous dynamics that satisfy Assumption 8.2 almost surely reaches an equilibrium state in finite time.*

To prove the theorem, we model the network game as a Markov chain and show that it is absorbing [55].

Proof of Theorem 8.3. Since the updates to $x(k+1)$ depend only on the state $x(k)$, and since agent activations do not depend on time, the network game can be modeled as a Markov chain with dimension 2^n. The state transition probabilities depend on the probabilities that each of the sets $\mathcal{A}_k$ will occur, along with the corresponding update dynamics. To prove almost sure convergence of the network game, it suffices to show that this Markov chain is *absorbing*, which requires satisfying two conditions [55, Definition 11.1, p416]. The first condition is that there exists at least one absorbing state. Absorbing states are equivalent to Nash equilibria of the network game, whose existence we have established in Corollaries 8.2 and 8.1. The second condition is that there exists a path in the Markov chain from every non-absorbing state to an absorbing state. Theorems 8.2 and 8.1 established the existence of such paths, which consist of finite sequences of asynchronous updates. It follows from Assumption 8.2, i.e., agent updates are independent and have support on $\mathbb{R}_{\geq 0}$, that the probabilities of each agent being the only active agent in a given time step are strictly positive (they can be computed from the probability distributions for the inter-activation times of each agent). Therefore, both conditions are met, and the Markov

chain is absorbing, which implies that the corresponding network game will almost surely reach an equilibrium state in finite time [55, Theorem 11.3, p417]. □

Note that one technically equivalent but perhaps less practical modeling of the partially synchronous updates would be to simply assume that at each time step $k = 0, 1, \ldots$, a random number of random agents activate simultaneously and then update at $k+1$. In other words, there is a fixed probability that a particular group of agents activate simultaneously at every time step. Then Theorem 8.3 still holds since the probability that each agent is activated asynchronously is bounded below by a positive constant. From a broader point of view, Theorem 8.3 holds whenever random asynchronous activations are not completely excluded from the partially synchronous dynamics. Convergence, however, may not be achieved for particular nonrandom activation sequences, discussed in the following.

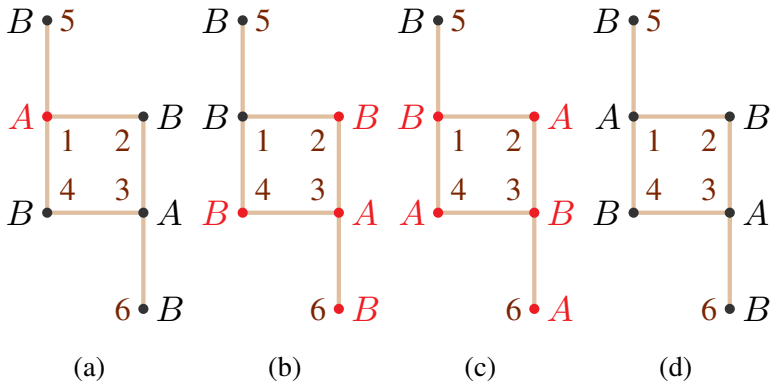


Figure 8.3: Strategy evolution in the network game $\Gamma = (\mathcal{G}, \frac{1}{2}\mathbf{1}, +)$ under the activation subsequence $\alpha_1 = (\{1\}, \{2, 3, 4, 6\}, \{1, 2, 3, 4, 6\})$. Red circles indicate those nodes that are activated at the corresponding time step.

8.4.3 Zero-probability non-convergence

We now show that Theorem 8.3 holds only almost surely. In other words, there exist activation sequences generated under Assumption 8.2 that do not result in equilibrium convergence; however, the probability of such sequences happening is zero. Consider the network game $\Gamma = (\mathcal{G}, \frac{1}{2}\mathbf{1}, +)$ where $\mathcal{G}$ and the initial strategies $x(0)$ are as in Figure 8.3-a. First, we show that if any single agent is activated when the strategy state equals $x(0)$, then there exists a finite sequence of agent activations that return the state to $x(0)$. Consider the case when agent 1 is activated exclusively at $k = 0$ (Figure 8.3-a), and hence, switches to B at $k = 1$. Then if agents 2, 3, 4, 6 activate at $k = 1$ and agents 1, 2, 3, 4, 6 at $k = 2$, the strategy state returns to $x(0)$ at $k = 3$, i.e., $x(3) = x(0)$; the process is shown in Figure 8.3-b to 8.3-d. Denote the corresponding activation subsequence by $\alpha_1 = (\{1\}, \{2, 3, 4, 6\}, \{1, 2, 3, 4, 6\})$. Similarly, as shown in Figure 8.4, there exists an activation sequence that returns the state of the system to $x(0)$ when agent 2 is activated at $k = 0$. The corresponding activation subsequence would be $\alpha_2 = (\{2\}, \{1, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\})$. Moreover, due to the symmetric distribution of the strategies $x(0)$ in the network, the same can

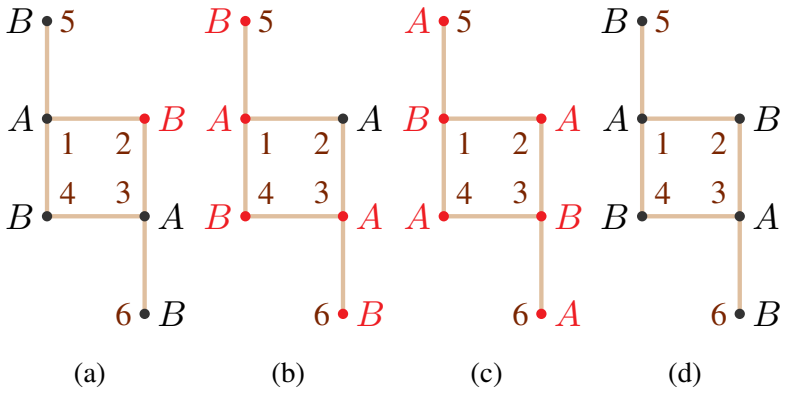


Figure 8.4: Strategy evolution in the network game $\Gamma = (\mathcal{G}, \frac{1}{2}\mathbf{1}, +)$ under the activation subsequence $\alpha_2 = (\{2\}, \{1,3,4,5,6\}, \{1,2,3,4,5,6\})$.

be shown for when agents 3, 4 or 6 are activated at $k = 0$ (which are similar to agents 1, 2 and 5, respectively). Denote the corresponding activation subsequences by α_3, α_4 and α_6. Now consider the event X made by α_1 and α_2 as follows

$$X = (x_i)_{i=1}^{\infty}, \quad x_i \in \{\alpha_1, \alpha_2\}.$$

Any activation sequence in X can be generated under Assumption 8.2 and has the property that all agents activate exclusively infinitely many times. However, no sequence in X results in convergence of the network game Γ. Yet this does not contradict Theorem 8.3 since $\mathcal{P}[X] = 0$, under Assumption 8.2.

8.5 Concluding remarks

We have shown that arbitrary networks consisting of all coordinating or all anti-coordinating agents who update with asynchronous best responses will reach equilibrium in finite time. Moreover, when updates are partially synchronous, we have shown that the network still almost surely reaches an equilibrium under mild conditions on the independence and randomness of agent updates. For the case of anti-coordinating agents, these results have important implications in social contexts where individuals prefer an action only if a small enough portion of neighbors are using that action, e.g., deciding which route to take to avoid traffic congestion, volunteering for a dangerous but important public service position, contributing money or time towards a crowd-sourced project, etc. For coordinating agents, the results apply to social contexts where each agents prefer an action only if a sufficient number of neighbors are using that action, e.g., the spread of social behaviors, technological innovations, viral infections. Our results suggest that in both cases, no matter how different the individuals are, which neighbors affect their decisions, or how many simultaneous decisions are made, everyone will tend to settle on a particular action with which they are satisfied. This means that the presence of cycles or non-convergence must result from other factors such as imitation or other unmodeled effects in the update dynamics, or a mixture of coordinating and anti-coordinating agents. These results also open the door to characterizing the equilibria and investigating possibilities for payoff-based incentive control of the network.

Chapter 9

Control of asynchronous best-response dynamics on networks through payoff incentives

Various collective behaviors occurring on networks of interacting agents can be modeled by asynchronous best-response dynamics, or equivalently, linear threshold dynamics. Building upon recent convergence results in this framework, we consider how such a network can be efficiently driven to a desired equilibrium state by offering payoff incentives or rewards for using a particular strategy, either uniformly or targeted to individuals. We begin by establishing the monotonicity of strategy changes following an increase in payoffs in coordination games, as well as the uniqueness of the resulting equilibria. Based on these results, for the case when a uniform incentive is offered to all agents, we show how to compute the optimal incentive using a binary search algorithm. When different incentives can be offered to each agent, we propose a new algorithm to select which agents should be targeted based on maximizing a ratio between a potential function measuring the effect on the network of an agent switching strategies and the incentive required to cause the agent to switch. Simulations show that this algorithm computes near-optimal targeted incentives for a wide range of networks and payoff distributions in coordination games and can also be effective for anti-coordination games.

9.1 Introduction

We use the convergence results in the previous chapter to provide control mechanisms for finite structured populations governed by the best response dynamics in this chapter. Faced with the rapidly growing scale and complexity of networked multi-agent systems, in which agents often have different and even competing objectives, control engineers are increasingly looking to game theory for answers to questions on convergence, stability, performance, robustness, distributed control and optimization, in various game contexts, e.g., potential games [97, 39, 89, 98, 90], stochastic games [10, 32, 182], matrix games [26], repeated games [96, 155], networked games [58], and others [47, 162, 11, 173, 95, 113, 51]. In particular, evolutionary game theory has proven to be a powerful tool towards investigating the dynamics and possibilities for control in large populations of

interacting decision-making agents [115, 138, 33, 91, 189]. The *best-response update rule*, in which agents choose the strategy that maximizes their total utility against the current strategies of their neighbors, is one of the simple yet intelligent mechanisms that evolutionary game theory postulates to understand the emergence of collective behaviors on networks of interacting individuals [149, 135, 139]. Two common social contexts capturing such behaviors are *coordination games*, where the adoption of a particular action by the neighbors of an individual stimulates the individual to adopt the action as well, such as in the spread of social innovations and viral infections, and *anti-coordination games*, where the adoption of a particular action by too many of the neighbors discourages the individual from adopting that action, such as in traffic congestion and the division of labor [37]. We refer to agents whose payoffs correspond to the above games as *coordinating* and *anti-coordinating*. In either context, the agents may make their decisions simultaneously, resulting in a *synchronous* update rule [2], or they may make decisions on independent time lines, resulting in an *asynchronous* update rule [187], which is particularly suitable when the rewards and consequences of the decisions take place more frequently than the decisions themselves. Several studies have investigated convergence in best-response dynamics for coordination and anti-coordination games in homogeneous populations, that is, when the utility functions of the individuals are the same, both on well-mixed populations [7] and networks [110, 88, 92], and some others have studied the more general heterogeneous case [2, 53, 136], where each individual has a possibly unique utility function. In particular, we have shown in Chapter 8 that every network consisting of either all coordinating or all anti-coordinating agents who update asynchronously with best responses will eventually reach an equilibrium state.

Having established these convergence results, we are now interested in how to drive such a network to consensus in a desired strategy by offering payoff incentives to the agents, motivated by applications such as hastening the adoption of new technologies or socially/environmentally beneficial behaviors and controlling the spread of viral epidemics [109]. Indeed, this is a fast growing research area in which several different approaches are possible, including what to consider as the control input. For example, under imitative dynamics, the goal in [145, 144] is to find the minimum number of agents such that when these agents adopt a desired strategy, the rest of the agents in the network will follow. The input in this work is thus the strategies of the agents, but it leaves open the question of how to implement such strategy control. In the context of best-response dynamics, a natural mechanism for achieving strategy control is the use of payoff incentives. For instance, in [91], the payoffs of a stochastic snowdrift game are changed in order to shift the equilibrium to a more cooperative one. This type of mechanism is applicable to situations where a central regulating agency has the power to uniformly change the payoffs of all agents to incentivize them to play a particular strategy. We refer to this control problem as *uniform reward control* where the goal is to lead individuals' to a desired strategy by offering the minimum uniform incentive to play that strategy. On the other hand, if the central agency can offer different rewards to each agent, a more efficient control protocol may be possible. That is, by altering the payoffs of just some individuals, the population can be led to a desired equilibrium state [144, 186]. We refer to this problem as *targeted reward control*. In case the budget for offering such rewards is limited, which may often be the case, a typical goal would be to maximize the number of individuals playing the desired strategy subject to the budget constraint, and we refer to this problem as *budgeted targeted reward control*.

In this chapter, we seek efficient incentive-based control algorithms for finite networks of hetero-

geneous decision-making individuals who asynchronously update their strategies to best responses, presented in the previous chapter. First, we prove that after increasing the rewards of a network of agents at equilibrium, who are all playing coordination games, the network almost surely converges to a unique equilibrium. This allows us to precisely predict the result of offering incentives to one or more agents under asynchronous best-response dynamics, which is in general not trivial since agents updating in random order can lead to many different outcomes. We use this property to provide efficient targeted-reward control protocols for both unlimited and limited budgets. In the case of uniform reward control, we use a binary search to find the optimal necessary reward. For targeted-reward control, we propose the Iterative Potential-to-Reward Optimization (IPRO) algorithm, which uses a threshold-based potential function and iteratively chooses the agent whose strategy switch maximizes the ratio of the increase in potential to the reward required to achieve the switch. We evaluate the performance of our protocol, by running several simulations and compare the results with those of some alternative approaches. Simulations on networks of coordinating agents show that the IPRO algorithm performs the best of those tested and near-optimal for a broad range of random networks and payoff distributions. For anti-coordinating agents, uniform and targeted reward control is trivial, yet budgeted targeted reward control remains challenging. Interestingly, our simulations suggest that if the potential decrease is weighted differently with respect to the rewards depending on the size of the available budget, the IPRO algorithm is also effective in this case.

9.2 Unique equilibrium convergence of coordinating network games

Our approach for reward-based control of the dynamics (8.2) in the previous chapter depends on some important convergence and monotonicity properties, for which we build upon our previous results proved in Chapter 8. Theorem 8.2 establishes convergence of asynchronous best-response dynamics on networks of coordinating agents, and requires only the weak assumption that each agent is activated infinitely many times as time goes to infinity. The theorem guarantees equilibrium convergence, leaving open the question of whether the equilibrium is unique. As the main theoretical result of this chapter, we show in the following that if the network starts from any equilibrium state, and the thresholds of some of the agents are decreased, the network almost surely reaches a *unique* equilibrium state, regardless of the sequence in which agents activate.

Theorem 9.1. *Consider a network of coordinating agents that is at equilibrium at time* 0*. If the thresholds of some of the agents are decreased at time* 0*, then almost surely after some finite time, the network will reach an equilibrium state which is unique and does not depend on the sequence in which agents activate.*

For the proof, we first show that under the condition of Theorem 9.1, the number of agents playing A evolves monotonically: when the network is at equilibrium, a decrease in one or more thresholds can only result in agents switching from B to A.

Proposition 9.1. *Consider a network of coordinating agents that is at equilibrium at time* 0*. If the thresholds of some of the agents are decreased at time* 0*, then no agent will switch from A to B at any time $t \geq 0$.*

Proof. The proof is done via contradiction. Assume the contrary and let $t_1 > 0$ denote the first time that some agent i switches from A to B. We know that the network was at equilibrium at time 0, so it follows from (8.2) that $n_i^A(0) > \tau_i \deg_i$. Since no thresholds are increased and node degrees are constant, the fact that agent i switched from A to B at time t_1 means that the number of A-neighbors of agent i at time $t_1 - 1$ must have been less than that at time 0, i.e., $n_i^A(t_1 - 1) < n_i^A(0)$. Therefore, at least one of the neighbors of agent i must have switched from A to B at some time before t_1, which contradicts how t_1 is defined, completing the proof. □

Next we show that after decreasing some of the thresholds in a network at equilibrium, any agents who switch from B to A under one activation sequence will do so under any activation sequence, although possibly at different times. Consider two activation sequences $\mathcal{S}^1 := \{i^0, i^1, \ldots\}$ and $\mathcal{S}^2 := \{j^0, j^1, \ldots\}$. Denote by $x_i^1(t)$ the strategy of agent i at time t under the activation sequence $\mathcal{S}^1$, and define $x_i^2(t)$ similarly for $\mathcal{S}^2$. Let t_0 be the first time when agent j^0 is active in $\mathcal{S}^1$. Then define t_s as the first time after t_{s-1} that agent j^s is active in $\mathcal{S}^1$, for $s \in \{1, 2, \ldots\}$. The existence of t_s is guaranteed by the assumption that each agent is activated infinitely many times.

Lemma 9.1. *Consider a network of coordinating agents that is at equilibrium at time* 0*. If the thresholds of some of the agents are decreased at time* 0*, then given any two activation sequences* $\mathcal{S}^1 = \{i^0, i^1, \ldots\}$ *and* $\mathcal{S}^2 = \{j^0, j^1, \ldots\}$*, the following holds for* $s \in \{0, 1, \ldots\}$*.*

$$x_{j^s}^2(s+1) = A \quad \Rightarrow \quad x_{j^s}^1(t_s + 1) = A. \tag{9.1}$$

Proof. The proof is via induction on s. First the statement is shown for $s = 0$. Suppose $x_{j^0}^2(1) = A$. If $x_{j^0}^2(0) = A$, i.e., agent j^0's strategy was already A in the beginning, then in view of Proposition 9.1, this agent will not switch to B regardless of the activation sequence. Hence, $x_{j^0}^1(t) = A$ for all $t \geq 0$, implying that (9.1) is in force. Next, assume that $x_{j^0}^2(0) = B$. Then agent j^0 has switched strategies at $t = 1$ under $\mathcal{S}^2$. Hence, in view of (8.2),

$$n_{j^0}^{A2}(0) \geq \tau'_{j^0} \deg_{j^0} \tag{9.2}$$

where τ'_i denotes the (possibly new) threshold of agent i after decreasing some thresholds at time 0 and $n_i^{A2}(t)$ denotes the number of A-playing neighbors of agent i at time t under the activation sequence $\mathcal{S}^2$. Similarly define $n_i^{A1}(t)$. Clearly

$$n_{j^0}^{A1}(0) = n_{j^0}^{A2}(0). \tag{9.3}$$

Due to Proposition 9.1, we also have

$$n_{j^0}^{A1}(t_0) \geq n_{j^0}^{A1}(0).$$

Hence, it follows from (9.3) that

$$n_{j^0}^{A1}(t_0) \geq n_{j^0}^{A2}(0).$$

Therefore, according to (9.2), $n_{j^0}^{A1}(t_0) \geq \tau'_{j^0} \deg_{j^0}$, implying that $x_{j^0}^1(t_0 + 1) = A$, which proves (9.1) for $s = 0$. Now assume that (9.1) holds for $s = 0, 1, \ldots, r - 1$. Similar to the case of $s = 0$, the induction statement can be proven for $s = r$: Suppose $x_{j^r}^2(r + 1) = A$. If $x_{j^r}^2(r) = A$, then according to Proposition 9.1, agent j^r will not switch to B regardless of the activation sequence.

Hence, $x^1_{j^r}(t) = A$ for all $t \geq r$, implying that (9.1) is in force for $s = r$. So assume that $x^2_{j^r}(r) = B$. Then agent j^r switches strategies at $t = r + 1$ under $\mathcal{S}^2$. Hence, in view of (8.2),

$$n^{A2}_{j^r}(r) \geq \tau'_{j^r} \deg_{j^r} . \tag{9.4}$$

Since (9.1) holds for all $s = 0, 1, \ldots, r - 1$, and because of Proposition 9.1, we obtain

$$n^{A1}_{j^r}(t_{r-1} + 1) \geq n^{A2}_{j^r}(r). \tag{9.5}$$

On the other hand, in view of Proposition 9.1, since $t_r \geq t_{r-1} + 1$,

$$n^{A1}_{j^r}(t_r) \geq n^{A1}_{j^r}(t_{r-1} + 1).$$

So because of (9.5), we get

$$n^{A1}_{j^r}(t_r) \geq n^{A2}_{j^r}(r).$$

Therefore, according to (9.4), $n^{A1}_{j^r}(t_r) \geq \tau'_{j^r} \deg_{j^r}$, implying that $x^1_{j^r}(t_r + 1) = A$, which proves (9.1) for $s = r$, completing the proof. □

We finally prove Theorem 9.1 by using Lemma 9.1 and Proposition 9.1.

Proof of Theorem 9.1. From Theorem 8.2, we know that regardless of the activation sequence, the network almost surely reaches an equilibrium state. So it remains to prove the uniqueness of the equilibrium for all activation sequences, which we do by contradiction. Assume that there exist two activation sequences $\mathcal{S}^1 = \{i^0, i^1, \ldots\}$ and $\mathcal{S}^2 = \{j^0, j^1, \ldots\}$ that almost surely drive the network to two distinct equilibrium states, implying the existence of an agent q whose strategy is different at the two equilibria, say B under the equilibrium of $\mathcal{S}^1$ and A under the equilibrium of $\mathcal{S}^2$. Hence, there exists some time τ after which the strategy of agent q is A under $\mathcal{S}^2$. So since each agent is active infinitely many times, there is some time $s \geq \tau$ at which agent q is active and plays strategy A at time $s+1$ under $\mathcal{S}^2$, i.e., $x^2_q(s+1) = A$. Then in view of (9.1) in Lemma 9.1, $x^1_q(t_s+1) = A$, that is the strategy of agent q becomes A at $t_s + 1$. On the other hand, according to Proposition 9.1, the strategy of agent q will not change after $t_s + 1$, i.e., $x^1_q(t) = A$ for all $t \geq t_s + 1$. But this is in contradiction with the assumption that the strategy of agent q is B at the equilibrium state under $\mathcal{S}^1$, completing the proof. □

9.3 Control through payoff incentives

In this section we consider the use of payoff incentives to drive a network of agents who update asynchronously with best responses from any undesired equilibrium toward a desired equilibrium in which all or at least more agents play strategy A.

9.3.1 Uniform reward control

Suppose a central regulating agency has the ability to provide a reward of $r_0 \geq 0$ to all agents who play strategy A. The resulting payoff matrix is given by

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & \quad B \end{array} \\ \begin{pmatrix} a_i + r_0 & b_i + r_0 \\ c_i & d_i \end{pmatrix} \end{array}, \qquad a_i, b_i, c_i, d_i \in \mathbb{R},$$

for each agent $i \in \mathcal{V}$. The control objective in this case is the following.

Problem 9.1 (Uniform reward control)**.** *Given a network game $\Gamma = (\mathcal{G}, \tau, \pm)$ and initial strategies $x(0)$, find the infimum reward r_0^* such that for every $r_0 > r_0^*$, $x_i(t)$ will almost surely reach A for every agent $i \in \mathcal{V}$.*

First, we observe that the solution to Problem 9.1 for networks of anti-coordinating agents is simply to choose r_0^* such that the thresholds of all agents are greater than or equal to one. For networks of coordinating agents, we first investigate how the agents' thresholds are affected by the reward.

Proposition 9.2. *If a coordinating agent i receives a positive reward for playing A, then the corresponding threshold will not increase, i.e., $\Delta\tau_i \leq 0$.*

Proof. First, we consider a non-stubborn coordinating agent, i.e., $\delta_i > 0$. The original threshold for such an agent is given by

$$\tau_i = \frac{\gamma_i}{\delta_i} = \frac{d_i - b_i}{a_i - c_i + d_i - b_i}.$$

After adding the reward, the new threshold is

$$\tau_i' = \frac{d_i - b_i - r_0}{a_i - c_i + d_i - b_i} = \tau_i + \Delta\tau_i,$$

where the change in threshold is given by

$$\Delta\tau_i = \frac{-r_0}{\delta_i}. \tag{9.6}$$

Hence, $\delta_i > 0$ implies $\Delta\tau_i \leq 0$. Next, we consider a stubborn coordinating agent, that is $\delta_i = 0$ and $\tau_i = 0$ if the agent is biased to A, and $\tau_i = 1$ if it is biased to B. Such an agent remains stubborn after adding any reward r_0. In particular, if the threshold of the agent is already 0, then the reward has no effect since the agent will still be biased to A. The threshold will also remain unchanged if it is originally 1, and the added reward is not enough to bias the agent to A. Otherwise, the reward changes the bias of the stubborn agent from B to A, making the threshold change from 1 to 0. Therefore, the change in threshold of a stubborn agent i is either 0 or -1, resulting in $\Delta\tau_i \leq 0$, which completes the proof. □

To compute the value of r_0^* for networks of coordinating agents, we take advantage of the following key properties of the dynamics: (i) the number of agents who converge to A is monotone in the value of r_0 due to Propositions 9.1 and 9.2, and (ii) simulations of the network game are fast to compute due to the unique equilibrium property established in Theorem 9.1. In other words, property (ii) means that since all activation sequences will result in the same equilibrium, we can choose a sequence consisting of only agents whose thresholds are violated, which will have a maximum length of n before reaching equilibrium. We begin by generating a set $\mathcal{R}$ of candidate infimum rewards defined by

$$\mathcal{R} \triangleq \left\{ r \geq \gamma_{\max} \,\middle|\, r = \frac{\delta_i(\check{n}_i^A - j)}{\deg_i}, i \in \mathcal{V}, j \in \{1, \ldots, \check{n}_i^A\} \right\}.$$

where

$$\gamma_{\max} = \begin{cases} \max\limits_{i \in \bar{\mathcal{B}}} \gamma_i & \bar{\mathcal{B}} \neq \emptyset \\ 0 & \bar{\mathcal{B}} = \emptyset \end{cases}$$

and $\bar{\mathcal{B}} = \{i \,|\, \delta_i = 0,\ x_i(0) = B\}$ is the set of stubborn agents biased to B. The set $\mathcal{R}$ is clearly finite, and indeed includes the optimal reward as shown in the following.

Proposition 9.3. *For a network of coordinating agents, $r_0^* \in \mathcal{R}$.*

Proof. According to Proposition 9.2, $\Delta\tau_i \leq 0$ for all $i \in \mathcal{V}$. So in view of Theorem 9.1, after adding a reward $r_0 > r_0^*$, the network almost surely reaches a unique equilibrium where everyone plays A, at some time t_f. For stubborn agents, we know that if they initially play A, they will keep doing so, and hence do not require a reward. However, if a stubborn agent is initially playing B, then in view of (8.1), the necessary and sufficient condition on the reward r_0 to make a stubborn agent i play A is $r_0 > \gamma_i$. Hence, $r_0^* \geq \gamma_i$, implying that r_0^* must be greater than $\gamma_{\max}$. On the other hand, in view of the update rule (8.2), to have all non-stubborn agents i play A, r_0^* must make the new thresholds τ_i' satisfy $n_i^A(t_f) \geq \tau_i' \deg_i$. Hence,

$$\begin{aligned} r_0^* &= \inf\left\{ r \geq \gamma_{\max} \,\middle|\, n_i^A(t_f) \geq (\tau_i - \tfrac{r}{\delta_i}) \deg_i \ \ \forall i \in \mathcal{V} \right\} \\ &= \inf\left\{ r \geq \gamma_{\max} \,\middle|\, r \geq \tfrac{\delta_i \left(\tau_i \deg_i - n_i^A(t_f)\right)}{\deg_i} \ \ \forall i \in \mathcal{V} \right\}. \end{aligned}$$

By definition, $\check{n}_i^A \leq \tau_i d_i + 1$ for all $i \in \mathcal{V}$. Hence,

$$\begin{aligned} r_0^* &= \inf\left\{ r \geq \gamma_{\max} \,\middle|\, r \geq \tfrac{\delta_i \left(\check{n}_i^A - (n_i^A(t_f)+1)\right)}{\deg_i} \ \ \forall i \in \mathcal{V} \right\} \\ &= \inf\left\{ r \geq \gamma_{\max} \,\middle|\, r = \tfrac{\delta_i \left(\check{n}_i^A - (n_i^A(t_f)+1)\right)}{\deg_i}, i \in \mathcal{V} \right\}. \end{aligned}$$

On the other hand, $n_i^A(t) \in \{0, 1, \ldots, \deg_i\}$ for all t and $i \in \mathcal{V}$, implying that

$$\begin{aligned} r_0^* &\in \left\{ r \geq \gamma_{\max} \,\middle|\, r = \tfrac{\delta_i \left(\check{n}_i^A - j\right)}{\deg_i}, i \in \mathcal{V}, j \in \{1, \ldots, \deg_i\} \right\} \\ &= \left\{ r \geq \gamma_{\max} \,\middle|\, r = \tfrac{\delta_i \left(\check{n}_i^A - j\right)}{\deg_i}, i \in \mathcal{V}, j \in \{1, \ldots, \check{n}_i^A\} \right\} \\ &= \mathcal{R}, \end{aligned}$$

which completes the proof. □

Let $v^{\mathcal{R}}$ denote the vector containing the elements of $\mathcal{R}$ sorted from lowest to highest. Algorithm 1 uses the fact that convergence of the network is monotone in the reward r_0 and performs a binary search to find the minimum candidate reward that results in all agents almost surely reaching strategy A. In what follows, $\mathbf{1}$ denotes the n-dimensional vector containing all ones.

Proposition 9.4. *Algorithm 1 computes the reward r_0^* that solves Problem 9.1 and terminates in $O(n \log |\mathcal{E}|)$ steps.*

$i^- := 1$
$i^+ := |\mathcal{R}|$
while $i^+ - i^- > 1$ **do**
  $r_0^* := v_j^{\mathcal{R}}$, where $j := \lceil \frac{i^- + i^+}{2} \rceil$
  $\Gamma' := (\mathcal{G}, \tau + \Delta\tau \mathbf{1}, +)$
  Simulate Γ' from $x(0)$ until equilibrium $\bar{x}$
  if $\bar{x}_i = A$ *for all* $i \in \mathcal{V}$ **then**
    $i^+ := j$
  else
    $i^- := j$
  end
end

Algorithm 1: Binary search algorithm to compute the reward r_0^* that solves Problem 9.1 for networks of coordinating agents.

Proof. Since $r_0^* \in \mathcal{R}$ due to Proposition 9.3, the minimum $r_0 \in \mathcal{R}$ which results in all agents switching to A is r_0^*. According to Theorem 9.1, if a given r_0 results in all agents switching to A for one activation sequence, then it does for every activation sequence. Therefore, we can test any given r_0 by activating only those agents whose thresholds are violated. Since agents can only switch from B to A after a decrease in thresholds, such a simulation requires no more than n activations. Due to Propositions 9.1 and 9.2, the number of agents switching to A is monotone in r_0, which means we can perform a binary search on the ordered list $v^{\mathcal{R}}$. Since the maximum number of elements in the set $\mathcal{R}$ is equal to the sum of the degrees of all nodes in the network which is equal to $2|\mathcal{E}|$, a binary search on $v^{\mathcal{R}}$ will result in $O(\log |\mathcal{E}|)$ iterations of the loop in Algorithm 1. The algorithm performs one simulation per iteration, and therefore requires $O(n \log |\mathcal{E}|)$ operations in total. □

9.3.2 Targeted reward control

If one has the ability to offer a different reward to each agent, it may be possible to achieve a desired outcome at a lower cost than with uniform rewards in networks of coordinating agents. This is because a small number of agents switching strategies can start a cascading effect in the network. Also, in a network with irregular topology and where the agents have different payoffs, some agents will generally require a smaller reward than others in order to adopt the desired strategy. Let $r := (r_1, \ldots, r_n)^T$ denote the vector of rewards offered to each agent, where r_i is the reward to agent i. We now have the following payoff matrix for each agent $i \in \mathcal{V}$:

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} a_i + r_i & b_i + r_i \\ c_i & d_i \end{pmatrix} \end{array}, \quad a_i, b_i, c_i, d_i \in \mathbb{R},\ r_i \in \mathbb{R}_{\geq 0}.$$

The targeted control objective is the following.

Problem 9.2 (Targeted reward control)**.** *Given a network game $\Gamma = (\mathcal{G}, \tau, \pm)$ and initial strategies $x(0)$, find the targeted reward vector r^* that minimizes $\sum_{i\in\mathcal{V}} r_i^*$ such that if $r_i > r_i^*$ for each i, then $x_i(t)$ will converge almost surely to A for every agent $i \in \mathcal{V}$.*

The solution to Problem 9.2 for networks of anti-coordinating agents is simply to set the threshold of every agent greater than or equal to one. Now consider a network of coordinating agents, which is at equilibrium at some time t_e. Let $\check{r}_i$ denote the infimum reward required for an agent playing B in this network to switch to A, which must satisfy the following according to (8.1):

$$\begin{aligned} &\delta_i n_i^A(t_e) = (\gamma_i - \check{r}_i)\deg_i \\ \Rightarrow \quad &\check{r}_i = \gamma_i - \frac{\delta_i n_i^A(t_e)}{\deg_i}. \end{aligned} \tag{9.7}$$

The corresponding new threshold is $\tau_i' = \tau_i + \Delta\tau_i$, where

$$\Delta\tau_i = \begin{cases} \dfrac{-\check{r}_i}{\delta_i} & \text{if } \delta_i \neq 0 \\ 0 & \text{if } \delta_i = 0 \wedge \gamma_i \leq 0 \\ -1 & \text{if } \delta_i = 0 \wedge \gamma_i > 0 \end{cases}.$$

In order to identify which agents should be offered incentives, we propose a potential function, which is a modification of the one used in Chapter 8 to prove convergence. Define the function $\Phi(x(t)) = \sum_{i=1}^n \Phi_i(x_i(t))$, where

$$\Phi_i(x(t)) = \begin{cases} n_i^A(t) - \check{n}_i^A(t) & \text{if } x_i(t) = A \\ n_i^A(t) - \check{n}_i^A(t) - 1 & \text{if } x_i(t) = B \end{cases}. \tag{9.8}$$

This function has a unique maximum, which occurs when all agents play A, and increases whenever an agent switches from B to A.

To evaluate the resulting change in the potential function $\Phi(x)$, we again use Theorem 9.1, which means that the network will reach a unique equilibrium and simulations are thus fast to compute using an activation sequence of length at most n. Denote this unique equilibrium by $\bar{x}$. The total change is then given by $\Delta\Phi(\bar{x}) := \Phi(\bar{x}) - \Phi(x(0))$. Let $\mathbf{e}_i$ denote the ith column of the $n \times n$ identity matrix. We propose Algorithm 2 to compute rewards based on iteratively choosing the uncontrolled B-agent that maximizes the ratio $\frac{\Delta\Phi(\bar{x})^\alpha}{\check{r}_i^\beta}$, where the exponents $\alpha \geq 0$ and $\beta \geq 0$ are degrees of freedom for the control designer, which we will explore further in the simulations section. Since $\check{r}_i$ is an infimum reward, we add an arbitrarily small amount ϵ to any nonzero reward r_i to ensure that the targeted agent will switch to A.

9.3.3 Budgeted targeted reward control

It is quite likely that any agency that wishes to influence a network of agents through the use of rewards has a limited budget with which to do so. In this section, we pose the following dual problem to the one in the previous section.

Initialize $\bar{x}_i = x_i(0)$ and $r_i = 0$ for each $i \in \mathcal{V}$
while $\exists i \in \mathcal{V} : \bar{x}_i \neq A$ *and* $\sum_{i \in \mathcal{V}} r_i < \rho$ **do**
  $\mathcal{B} := \{i \in \mathcal{V} : \bar{x}_i = B \wedge \check{r}_i \leq \rho - \sum_{i \in \mathcal{V}} r_i\}$
  foreach $i \in \mathcal{B}$ **do**
    Let $\Gamma_i = (\mathcal{G}, \tau + \Delta\tau_i \mathbf{e}_i, +)$
    Simulate Γ_i starting from $\bar{x}$ until next equilibrium
    Set $\bar{x} :=$ new equilibrium and compute $\Phi(\bar{x})$
  end
  $j = \arg\max_{i \in \mathcal{B}} \dfrac{\Delta\Phi(\bar{x})^\alpha}{\check{r}_i^\beta}$
  $r_j := \check{r}_j + \epsilon$
  $\tau_j := \tau_j + \Delta\tau_j$
end

Algorithm 2: Approximates the solutions to Problems 9.2 and 9.3 for networks of coordinating agents by iteratively choosing the B-agent whose switch to A maximizes the ratio $\frac{\Delta\Phi(\bar{x})^\alpha}{\check{r}_i^\beta}$ and whose required reward lies within the budget constraint ρ, where $\rho := \infty$ when there is no budget constraint.

Problem 9.3 (Budgeted targeted reward control)**.** *Given a network game $\Gamma = (\mathcal{G}, \tau, \pm)$, initial strategy state $x(0)$, and budget constraint $\sum_{i \in \mathcal{V}} r_i < \rho$, find the reward vector r that maximizes the number of agents in the network who almost surely reach A.*

Algorithm 2 can also be used to approximate the solution to this problem. The only difference is that the algorithm will now terminate if no more agents can be incentivized to switch to A without violating the budget constraint ρ.

9.4 Simulations

In this section, we compare the performance of the proposed algorithm to some alternative approaches. Short descriptions of each algorithm are provided below. Each of these methods is applied iteratively, targeting agents until either the control objective is achieved or the budget limit is reached.

- **Iterative Random (rand):** target random agents in the network
- **Iterative Degree-Based (deg):** target agents with maximum (minimum) degree for networks of coordinating (anti-coordinating) agents
- **Iterative Potential Optimization (IPO):** target agents resulting in the maximum increase of the potential function ($\alpha = 1$, $\beta = 0$)
- **Iterative Reward Optimization (IRO):** target agents requiring minimum reward ($\alpha = 0$, $\beta = 1$)

- **Iterative Potential-to-Reward Optimization (IPRO):** target agents maximizing the potential-change-to-reward ratio ($\alpha > 0$, $\beta > 0$)

For each set of simulations, we generate geometric random networks by randomly distributing n agents in the unit square and connecting all pairs of agents who lie within a distance R of each other. We focus on the case when all agents are coordinating to align with our theoretical results, but we also include one simulation study on a network of anti-coordinating agents to show that the proposed algorithm can be applied to more general cases. In all simulations of the IPRO algorithm, we used $\alpha = 1$ and $\beta = 4$.

9.4.1 Uniform vs. targeted reward control

First, we investigate the difference between uniform and targeted reward control to estimate the expected cost savings when individual agents can be targeted for rewards rather than offering a uniform reward to all agents. Figure 9.1 shows not only that targeted reward control offers a large cost savings over uniform rewards, but that the savings increases with network size.

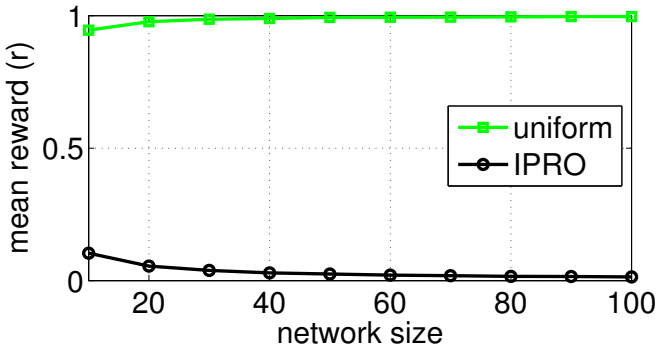


Figure 9.1: Comparison of uniform and targeted reward control on geometric random networks for a range of sizes. For each size tested, 500 random networks were generated using a connection radius $R = \sqrt{(1 + \deg_{\text{exp}})/\pi n}$, corresponding to a mean node degree of approximately $\deg_{\text{exp}} = 10$. Thresholds τ_i for each agent are uniformly randomly distributed on the interval $[0, \frac{2}{3}]$, and the corresponding payoffs are $a_i = \frac{1-\tau_i}{\tau_i}$, $b_i = c_i = 0$, and $d_i = 1$.

9.4.2 Targeted-reward control: network size

Next, we compare the performance of the proposed control algorithms to some alternative approaches for various sizes of networks of coordinating agents, using the same network and threshold setup as the previous section. Fig. 9.2 shows that the IPRO algorithm performs consistently better than the other proposed approaches across all network sizes, although the IRO method requires only slightly larger rewards on average than IPRO.

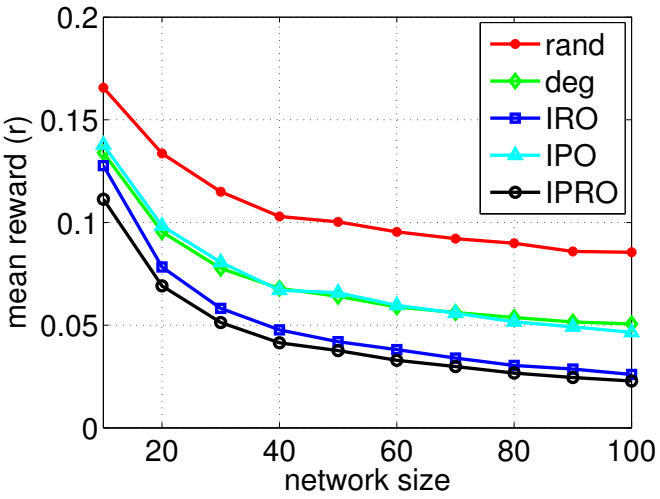


Figure 9.2: Comparison of algorithms for different sizes of networks. The connection radius, threshold distribution, and payoffs are generated exactly as in the simulations for Fig. 9.1.

9.4.3 Targeted-reward control: network connectivity

We now investigate how the connectivity of a network affects the reward needed to achieve consensus in strategy A. We consider geometric random networks of only 12 agents, which is small enough that we can compare against the true optimal solution computed using an exhaustive search algorithm. Fig. 9.3 shows that there appears to be a transition region in the required reward between sparsely and densely connected networks, and we see that the IPRO algorithm yields near-optimal results across the entire range, while the IRO algorithm also performs quite well for dense networks.

9.4.4 Targeted-reward control: threshold level

In this section, we investigate the performance of various algorithms as the thresholds of agents increase and thus become more costly to control. We again consider geometric random networks of only 12 agents and thresholds of no greater than 0.5 in order to compare against the optimal solution. Fig. 9.4 shows that the IPRO algorithm maintains the best performance across this range of threshold values, while the distance from optimality increases slightly as the mean threshold increases.

9.4.5 Targeted-reward control: threshold variance

In the next set of simulations, we change the threshold variance to understand the effect of increasing heterogeneity on the performance of the algorithms. Fig. 9.5 shows that the IPRO algorithm again performs the best of the alternative algorithms. Moreover, as the threshold variance increases, its performance approaches that of the optimal solution.

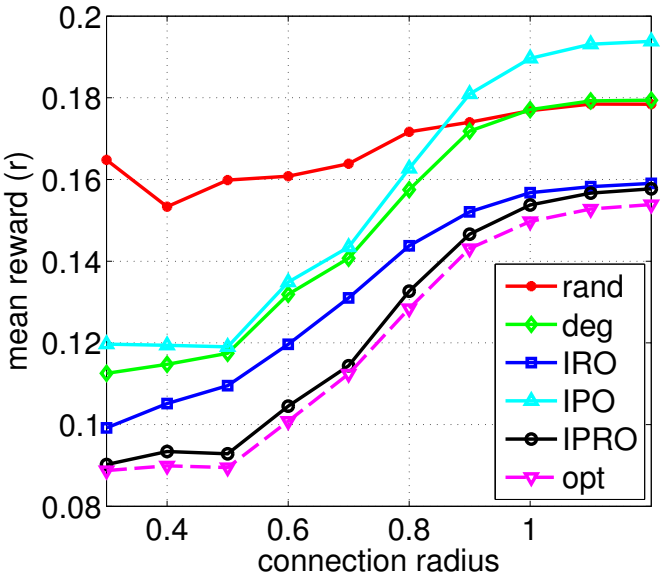


Figure 9.3: Comparison of algorithms on sparsely to densely connected 12-node networks. 100 networks are tested for each connection range, and the threshold distribution and payoffs are generated exactly as in the simulations for Fig. 9.1.

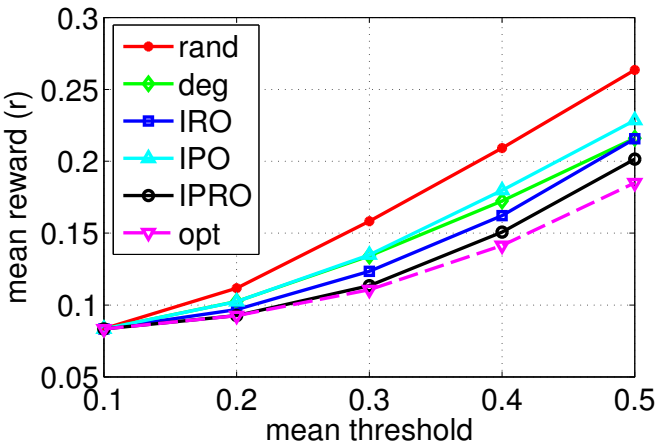


Figure 9.4: Comparison of algorithms for various mean thresholds of coordinating agents. 500 12-node networks are tested for each mean threshold value τ_0, and the connection radius R is drawn uniformly at random from the interval $[0.3, 1]$. Agent thresholds are uniformly distributed on the interval $\tau_0 \pm 0.1$.

9.4.6 Budgeted targeted reward control

Finally, we consider the case when there is a limited budget from which to offer rewards. Figures 9.6 and 9.7 show the results for the cases of coordination and anti-coordination, respectively. In the coordination case, we see that IPRO achieves greater convergence to A at lower costs when compared to the other approaches. Interestingly, the IPO algorithm also performs quite well for low-budget cases. However, there remains significant sub-optimality of all approaches in the low

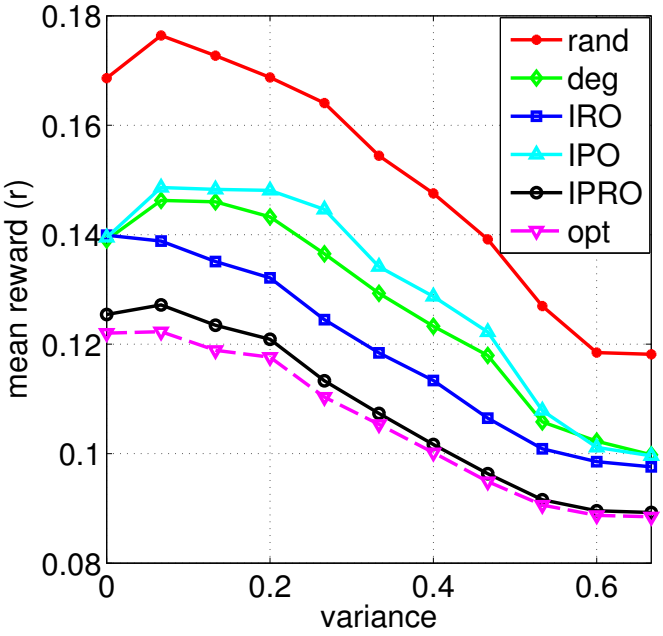


Figure 9.5: Comparison of algorithms for different threshold variances w. 500 12-node networks are tested for each value of w and the thresholds are uniformly randomly distributed in the interval $\frac{1}{3} \pm \frac{w}{2}$.

to middle range of reward budgets. Since budgeted targeted reward control is the only problem

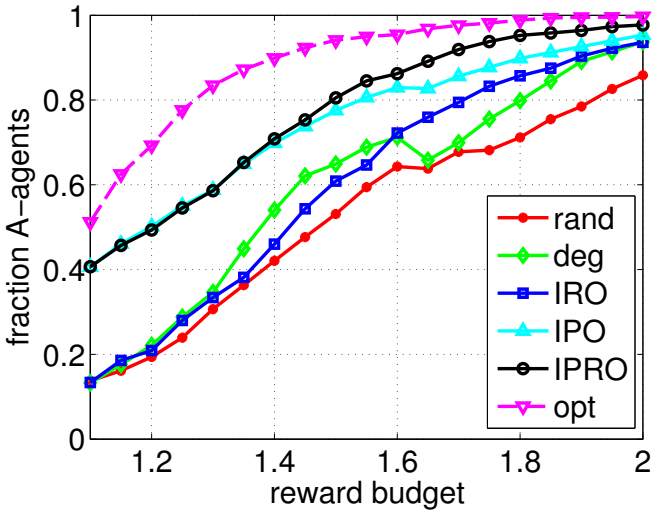


Figure 9.6: Comparison of algorithms for budgeted targeted reward control on networks of coordinating agents for a range of reward budgets. 500 networks were tested with 50-nodes each and a connection range $R = 0.2$. Thresholds are uniformly distributed on the interval 0.5 ± 0.1.

that has a nontrivial solution for anti-coordinating agents, we also compared the algorithms for an anti-coordinating case. Here, we observe that while IRO works best for small reward budgets,

IPO performs best for larger reward budgets. This suggests setting the exponent α small for low budgets and large for high budgets while doing exactly the opposite for the exponent β.

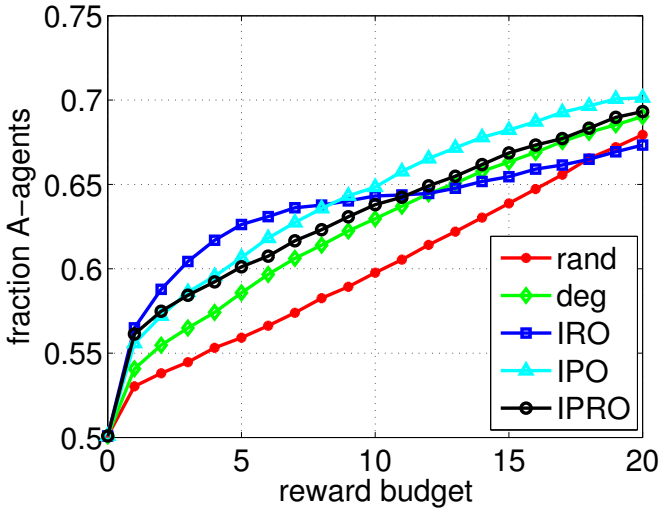


Figure 9.7: Comparing the performance of several algorithms on the budgeted targeted reward control problem for anti-coordination games on 50-node networks ($R = 0.2$). Thresholds are uniformly distributed on the interval 0.5 ± 0.1.

9.5 Concluding remarks

We have considered three problems related to the control of asynchronous best-response dynamics on networks through payoff incentives. Our proposed solutions are based on the following key theoretical results: (i) after offering rewards to some of the agents in a coordinating network which is at equilibrium, strategy switches occur only in one direction, and (ii) the network almost surely reaches a unique equilibrium state. When a central agency can only offer a uniform reward to all agents, the minimum value of the reward can be computed using a binary search algorithm and fast simulations, made possible by these monotonicity and uniqueness results. If rewards can be targeted towards individual agents, the desired convergence can be achieved at much lower cost; however, the problem becomes somewhat more complex to solve. To approximate the solution in this case, we propose the IPRO algorithm, which iteratively selects the agent who, upon switching strategies, maximizes the ratio between the resulting change in potential and the cost of achieving such a switch, until desired convergence is achieved. A slight modification of this algorithm applies to the case when the budget from which to offer rewards is limited. Simulations show that this algorithm performs significantly better than other algorithms based on threshold or degree, and in many cases very close to the true optimal solution.

Chapter 10

Imitating successful neighbors hinders reaching satisfactory decisions

Imitation is widely observed in nature and often used to model populations of decision-making agents, but it is not yet known under what conditions such a population will converge to a state in which the agents are satisfied with their decisions. Here we show that every network in which agents imitate the best performing strategy in their neighborhood will reach an equilibrium in finite time, provided that all agents are *opponent-coordinating*, i.e., earn a higher payoff if their opponent plays the same strategy as they do. It follows that any non-convergence observed in such networks is not a result of population heterogeneity nor irregular network topology, but rather must be caused by other factors such as the presence of non-opponent-coordinating agents. To strengthen this result, we show that large classes of networks containing other types of agents may never converge under imitation. Comparing to the best-response case, this implies that networks of imitating agents are less likely to settle on an equilibrium state.

10.1 Introduction

After studying finite structured populations of individuals updating according to the best-response update rule in Chapters 8 and 9, we focus on the case with the imitation update rule in this and the following chapter. In social, economic, biological, technological, and other types of networks, the dynamics of interconnected agents may give rise to complex and seemingly unpredictable behaviors. While some networks may converge to a state of equilibrium, others may perpetually cycle or experience chaotic fluctuations [163]. Unfortunately, localized analysis may reveal little about the underlying causes of these emergent behaviors, in part because the major factors driving the dynamics may lie not in the individual agents but in their complex interconnections. However, studying the system from a broader perspective, perhaps at the cost of substantial simplification of the agent-level dynamics, can help to characterize critical properties such as the conditions on the agents under which a network is likely to converge or not [149]. Indeed we have seen a substantial transition from local to network-based analysis across various disciplines in engineering and the physical and social sciences. This network-oriented approach has led to many influential discoveries of collective behaviors, even when the individual agents are autonomous and highly

complex. One of the primary toolsets invoked to study these kinds of problems is *evolutionary game theory* [67, 138, 139, 4].

Originally proposed as a framework to study behaviors such as ritualized fighting in animals [160], evolutionary game theory has since been widely adopted in the social sciences. A key innovation of evolutionary game theory is that rather than assuming agents make perfectly rational choices, strategies and behaviors propagate through a population via some dynamic process. In the biological world, this propagation is manifested through *survival of the fittest* and reproduction, which is widely modeled by the *replicator dynamics* [181]. This system of first-order differential equations provides a useful and simple framework for studying population dynamics, under the assumptions that populations are infinite and well-mixed. While these assumptions can lead to reasonable approximations for large, dense populations of organisms, there are many networks, especially those involving humans, in which the structure of the interactions plays a major role [117]. Fortunately, it is still possible to study replicator-like dynamics in populations connected by networks [121], and it turns out that certain models of imitation, where individuals mimic the strategy of their most successful neighbor, reduce exactly to the replicator dynamics in the limit of large networks [153]. An extensive literature has emerged in the field of evolutionary games on networks, especially regarding the question of how cooperation can evolve and persist under various conditions and in various types of networks [165, 142, 87, 126, 4, 102, 54]. In particular, networks of imitative individuals are reported to reach low levels of cooperation [176].

Another fundamental yet less-studied question for imitative dynamics on networks is whether each individual will settle on a particular action, or equivalently, whether the network will converge to an equilibrium. Several previous studies have investigated this problem in various contexts. For example, it was shown in [37] that even when agent dynamics are homogeneous (symmetric) and synchronous, imitation does not generally result in convergence to equilibria, although it appears likely for coordination games. Also in the synchronous and homogeneous setting, [185] characterizes a restrictive set of topological properties for which networks will converge to an equilibrium. It remains to be seen under what conditions arbitrary networks of heterogeneous agents updating synchronously or asynchronously can be expected to settle at an equilibrium, and that is the primary goal of this chapter.

The main contribution of this chapter is to provide sufficient conditions under which networks of imitating agents can be expected to reach an equilibrium. The key criterion turns out to be whether agents earn a higher payoff when their opponents play the same strategy as them. We call such agents *opponent-coordinating*, and indeed we show that every network consisting entirely of opponent-coordinating agents will reach an equilibrium under asynchronous updates. Although not equal, the set of payoff matrices corresponding to such agents has a significant intersection with that of coordination games, which explains why they are reported to be more likely to lead the network to an equilibrium. If the agents are strongly opponent-coordinating, then equilibrium convergence is guaranteed even when multiple agents update simultaneously at any time step, that is, when the updates are fully or partially synchronous. These results establish a strong set of sufficient conditions for convergence of a network of imitative agents, and also imply the presence of non-opponent coordinating agents in such networks when exhibiting non-convergent behavior. Knowing when a network is expected to converge also opens up avenues for further research into convergence rates and control. As the second contribution, we provide several examples on heterogeneous networks including non-opponent-coordinating agents that never converge to a single

state. The examples can be extended to substantial classes of networks of homogeneous agents. The comparison of these results with those of the best-response update rule, known to govern any network of homogeneous agents to an equilibrium [140], postulates convergence under imitation as a rare phenomenon, and also helps to explain why networks in which imitation is prevalent may exhibit cyclic or chaotic behavior more often than networks in which agents use best-response updates, for example.

10.2 Asynchronous imitation updates

Consider an undirected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the nodes $\mathcal{V} = \{1, \dots, n\}$ correspond to agents who over time $k \in \{0, 1, \dots\}$, play 2-player games with their neighbors indicated by the edges $\mathcal{E}$. The agents start with one of the strategies $1, 2, \dots, m$ at $k = 0$. At each time step, every agent earns an accumulated payoff against her neighbors, and then one random agent is activated to mimic the strategy of the most successful agent in her neighborhood. More specifically, the possible payoffs of an agent $i \in \mathcal{V}$ against another agent j are summarized in the *payoff matrix* $\pi^i \in \mathbb{R}^{m \times m}$ whose pq^{th} entry corresponds to the strategy pair p-against-q where $p, q \in \{1, \dots, m\}$. Let $x_i(k)$ denote the strategy of agent i at time k. Then the payoff of agent i at time k against her neighbors is calculated by

$$u_i(k) = \sum_{j \in \mathcal{N}_i} \pi^i_{x_i(k), x_j(k)}$$

where $\mathcal{N}_i$ denotes the set of neighbors of agent i. The imitation update rule for agent i, active at time k, dictates that agent i updates her strategy at $k+1$ to the strategy of the agent earning the highest payoff at k in her neighborhood $\mathcal{N}_i \cup \{i\}$. In case several agents with different strategies earn the highest payoff, we assume agent i sticks to her current strategy if she is also earning the highest payoff. Otherwise, we assume a preference on the strategies such that agent i chooses the smallest strategy in magnitude, namely

$$x_i(k+1) = \begin{cases} x_i(k) & x_i(k) \in \mathcal{S}_i^M(k) \\ \min \mathcal{S}_i^M(k) & x_i(k) \notin \mathcal{S}_i^M(k) \end{cases} \tag{10.1}$$

where $\mathcal{S}_i^M(k)$ is the set of strategies resulting in the maximum payoff at time k in the neighborhood of agent i, that is

$$\mathcal{S}_i^M(k) \triangleq \left\{ x_j(k) \,\middle|\, u_j(k) = \max_{r \in \mathcal{N}_i \cup \{i\}} u_r(k) \right\}.$$

The main goal of this chapter is to study the evolution of the strategy vector

$$x(k) = (x_1(k), \dots, x_n(k))^\top$$

, under Update rule (10.1) and the activation sequence of the agents, defining the asynchronous imitation dynamics. We assume that the activation sequence is *persistent*, that is for every agent $j \in \mathcal{V}$ and every time $k \geq 0$, there exists some finite later time $k' > k$ at which agent j is active. This mild condition is equivalent to saying that each agent is activated infinitely many times.

An equilibrium of the dynamics is a state x^* at which none of the agents will change strategies if activated, implying that if $x(k) = x^*$ for some $k \geq 0$, then $x(k+1) = x^*$, regardless of which

agent is active at k. Because of the implicit stochasticity caused by the activation sequence and the nonlinearity of the imitation dynamics, convergence of $x(k)$ to an equilibrium is not guaranteed. Indeed we provide examples where $x(k)$ fluctuates in a set of several states in the long run and never converges to a single state. However, there are types of payoff matrices for which the network always reaches an equilibrium, as we show in the next section.

10.3 Convergence under asynchronous updates

Unlike dynamics such as the best-response where convergence relies on the ordering of each column's entries of the payoff matrices (i.e., whether the agents are coordinating or anti-coordinating [140]), in imitation, the ordering of each row's entries can be more important. This is because under best-response, every agent finds the best available strategy to play against the current state of the network, whereas under imitation, agents never look at how different their payoffs will be if they change their strategies, yet switches of neighbors' strategies can change the agent with maximum payoff in a neighborhood. We call an agent $i \in \mathcal{V}$ a *opponent-coordinating* agent if each diagonal entry of her payoff matrix π^i is greater than all off-diagonal elements in the same row, that is

$$\pi^i_{p,p} > \pi^i_{p,q} \qquad \forall p, q \in \{1, \dots, m\}, p \neq q. \tag{10.2}$$

Intuitively, this condition implies that agent i's payoff increases, if a neighbor of agent i switches her strategy to that of agent i. Payoff matrices of stag-hunt and prisoners' dilemma games [149] may or may not satisfy the condition, however, those of snowdrift games never do. It turns out that convergence of imitation dynamics can be guaranteed for networks of opponent-coordinating agents, which we show in the first part of this section. Then we extend the result to when the network also contains non-opponent-coordinating agents whose strategies are ensured to become fixed after some amount of time.

10.3.1 Convergence of opponent-coordinating agents

Theorem 10.1. *Every network of opponent-coordinating agents reaches an equilibrium under the asynchronous imitation update rule.*

For the proof, we make use of a Lyapunov-like function $u_{\mathcal{M}}$, the payoff of the agent(s) with maximum payoff, formally defined in the following. Define $\mathcal{M}(k)$ as the set of agents with maximum payoff in the network at time k:

$$\mathcal{M}(k) = \left\{ i \,\middle|\, u_i(k) = \max_{j \in \mathcal{V}} u_j(k) \right\}.$$

Clearly $\mathcal{M}(k)$ is nonempty for all $k \geq 0$. At time k, every member of $\mathcal{M}(k)$ earns the same payoff that we denote by $u_{\mathcal{M}(k)} \in \mathbb{R}$:

$$u_{\mathcal{M}(k)} = u_i(k) \qquad \forall i \in \mathcal{M}(k).$$

Clearly $u_{\mathcal{M}(k)}$ is upper bounded. The following lemma guarantees that $u_{\mathcal{M}(k)}$ is always non-decreasing, and increases when any of the neighbors of any of the agents with the maximum payoff switches strategies.

Lemma 10.1. *The following holds*

$$u_{\mathcal{M}(k+1)} \geq u_{\mathcal{M}(k)} \qquad \forall k \geq 0. \tag{10.3}$$

Moreover, given $k \geq 0$, if a neighbor of an agent $i \in \mathcal{M}(k)$ switches strategies at $k+1$, then

$$u_{\mathcal{M}(k+1)} > u_{\mathcal{M}(k)}. \tag{10.4}$$

Proof. Consider an agent $i \in \mathcal{M}(k)$. This agent does not change her strategy at $k+1$ since she is already earning the maximum payoff in the network. In general, one of the following two cases takes place.

Case 1: None of the neighbors of agent i switch strategies at $k+1$. Then the payoff of agent i does not change, i.e.,

$$u_i(k+1) = u_i(k). \tag{10.5}$$

Now one of the following two cases can happen at $k+1$.

Case 1-1: $i \in \mathcal{M}(k+1)$. Then according to (10.5),

$$u_{\mathcal{M}(k+1)} = u_{\mathcal{M}(k)} \tag{10.6}$$

Case 1-2: $i \notin \mathcal{M}(k+1)$. Then there exists an agent $j \in \mathcal{M}(k+1)$ who earns more than agent i at $k+1$. This in view of (10.5) implies $u_j(k+1) > u_i(k)$, resulting in

$$u_{\mathcal{M}(k+1)} > u_{\mathcal{M}(k)}. \tag{10.7}$$

By combining (10.5) and (10.7), we arrive at (10.3).

Case 2: A neighbor r of agent i, switches strategies at $k+1$. According to update rule (1) in the main text, agent r changes her strategy to that of one of her neighbors, say j, that has the highest payoff among the rest (j may equal i). Hence, since agent i is already earning the highest payoff in the network at k, agent j must also do so, i.e., $j \in \mathcal{M}(k)$. Therefore, agent j does not switch strategies at $k+1$. Moreover, because of the asynchronous updating, no other neighbor of j except for agent r switches strategies at $k+1$. So, the payoff of agent j at k equals

$$u_j(k) = \pi^j_{x_j(k),x_r(k)} + \sum_{s \in \mathcal{N}_j - \{r\}} \pi^j_{x_j(k),x_s(k)},$$

and at $k+1$ equals

$$u_j(k+1) = \pi^j_{x_j(k),x_j(k)} + \sum_{s \in \mathcal{N}_j - \{r\}} \pi^j_{x_j(k),x_s(k)}.$$

Hence,

$$u_j(k+1) - u_j(k) = \pi^j_{x_j(k),x_j(k)} - \pi^j_{x_j(k),x_r(k)} \overset{(10.2)}{>} 0. \tag{10.8}$$

Now one of the following two cases can happen at $k+1$.

Case 2-1: $j \in \mathcal{M}(k+1)$. Then according to (10.17),

$$u_{\mathcal{M}(k+1)} > u_{\mathcal{M}(k)}. \tag{10.9}$$

Case 2-2: $j \notin \mathcal{M}(k+1)$. Then there exists an agent $q \in \mathcal{M}(k+1)$ who earns more than agent j at $k+1$. This in view of (10.17) implies $u_q(k+1) > u_j(k)$, resulting in

$$u_{\mathcal{M}(k+1)} > u_{\mathcal{M}(k)}. \tag{10.10}$$

By combining (10.9) and (10.10), we arrive at (10.4), which completes the proof. □

Using Lemma 10.1, we can show the convergence of the strategies of the agents with maximum payoffs.

Lemma 10.2. *There exists some time $k_1 \geq 0$ after which the strategies of all agents in $\mathcal{M}(k_1)$ and their neighbors remain unchanged, i.e.,*

$$x_j(k) = x_j(k_1) \qquad \forall j \in \mathcal{N}_i \cup \{i\},\ \forall i \in \mathcal{M}(k_1),\ \forall k \geq k_1. \tag{10.11}$$

Proof. Since $u_{\mathcal{M}(k)}$ is upper-bounded, Lemma 10.1 implies the existence of some time $k_{\mathcal{M}} \geq 0$, such that

$$u_{\mathcal{M}(k)} = u_{\mathcal{M}(k_{\mathcal{M}})} \qquad \forall k \geq k_{\mathcal{M}}, \tag{10.12}$$

and that for $k \geq k_{\mathcal{M}}$, none of the neighbors of any agent $i \in \mathcal{M}(k)$ switch strategies at $k+1$:

$$x_j(k) = x_j(k+1) \qquad \forall j \in \mathcal{N}_i\ \forall i \in \mathcal{M}(k),\ \forall k \geq k_{\mathcal{M}}. \tag{10.13}$$

Now (10.12) implies that if an agent earns the maximum payoff at some time $k \geq k_{\mathcal{M}}$, she will keep doing so in future time steps, yielding $\mathcal{M}(k) \subseteq M(k+1)$ for all $k \geq k_{\mathcal{M}}$. Therefore, since $\mathcal{M}$ is upper-bounded by $\mathcal{V}$, it will become fixed after some time $k_1 \geq k_{\mathcal{M}}$:

$$\mathcal{M}(k) = \mathcal{M}(k_1) \qquad \forall k \geq k_1.$$

Hence, in view of (10.13),

$$x_j(k) = x_j(k+1) \qquad \forall j \in \mathcal{N}_i\ \forall i \in \mathcal{M}(k_1),\ \forall k \geq k_1,$$

resulting in

$$x_j(k) = x_j(k_1) \qquad \forall j \in \mathcal{N}_i\ \forall i \in \mathcal{M}(k_1),\ \forall k \geq k_1.$$

Now since the strategies of all neighbors of each agent $i \in \mathcal{M}(k_1)$ are fixed, the strategy of agent i herself also becomes fixed in view of update rule (1) in the main text, which results in (10.11). □

Now consider the time k_1 in Lemma 10.2, and define the set

$$\mathcal{V}_2 = \mathcal{V} - \mathcal{M}(k_1).$$

If $\mathcal{V}_2$ is empty, the network has reached a state which has to be an equilibrium due to the persistent assumption on the activation sequence. Otherwise, define $\mathcal{M}_2$ similar to how $\mathcal{M}$ is defined, i.e.,

$$\mathcal{M}_2(k) = \left\{ i \,\middle|\, u_i(k) = \max_{j \in \mathcal{V}_2} u_j(k) \right\}.$$

Clearly $\mathcal{M}_2(k)$ is nonempty for all $k \geq k_1$. Denote by $u_{\mathcal{M}_2(k)}$ the payoff of any member in $\mathcal{M}_2(k)$ at time k:

$$u_{\mathcal{M}_2(k)} = u_i(k) \qquad \forall i \in \mathcal{M}_2(k).$$

One can prove a similar result to that of Lemma 10.1, for $u_{\mathcal{M}_2(k)}$, as shown in the following lemma.

Lemma 10.3. *Consider the time k_1 in Lemma 10.2. It holds that*

$$u_{\mathcal{M}_2(k+1)} \geq u_{\mathcal{M}_2(k)} \qquad \forall k \geq k_1. \tag{10.14}$$

Moreover, given $k \geq k_1$, if a neighbor of an agent $i \in \mathcal{M}_2(k)$ switches strategies at $k+1$, then

$$u_{\mathcal{M}_2(k+1)} > u_{\mathcal{M}_2(k)}. \tag{10.15}$$

Proof. Consider some time $k \geq k_1$ and an agent $i \in \mathcal{M}_2(k)$. If $i \in \mathcal{N}_j$ for some $j \in \mathcal{M}(k)$, then in view of Lemma 10.2, agent i does not change her strategy at $k+1$. Otherwise, agent i is not a neighbor of any agent in $\mathcal{M}(k)$. Then, since the only agents in the network who earn more than agent i at k are those in $\mathcal{M}(k)$, agent i earns more than all of her neighbors at time k, implying that again she does not change her strategy at $k+1$. Therefore, in any case, agent i does not switch strategies at $k+1$. Now one of the following two cases takes place.

Case 1: None of the neighbors of agent i switch strategies at $k+1$. Then similar to the proof of Lemma 10.2, we arrive at (10.14).

Case 2: A neighbor r of agent i, switches strategies at $k+1$. According to update rule (1) in the main text, agent r changes her strategy to that of one of her neighbors, say j, that has the highest payoff among the rest (j may equal i). Now in view of Lemma 10.2, agent r is not a neighbor of any agent in $\mathcal{M}(k)$. Hence, $j \notin \mathcal{M}(k)$. On the other hand, since agent i is a neighbor of agent r and is earning the second highest payoff in the network at time k, the same must hold for agent j, i.e., $j \in \mathcal{M}_2(k)$. The rest of the proof can be done similar to that of Lemma 10.1, by replacing $\mathcal{M}$ with $\mathcal{M}_2$. □

The following lemma can also be proven similar to Lemma 10.2, which guarantees the convergence of the strategies of agents with the second-maximum payoffs.

Lemma 10.4. *Consider the time k_1 in Lemma 10.2. There exists some time $k_2 \geq k_1$ after which the strategies of all agents in $\mathcal{M}_2(k_2)$ and their neighbors remain unchanged, i.e.,*

$$x_j(k) = x_j(k_1) \qquad \forall j \in \mathcal{N}_i \cup \{i\},\ \forall i \in \mathcal{M}_2(k_2),\ \forall k \geq k_2.$$

Now consider the time k_2 in Lemma 10.4, and define the set

$$\mathcal{V}_3 = \mathcal{V}_2 - \mathcal{M}_2(k_2).$$

If $\mathcal{V}_3$ is empty, the network has reached an equilibrium. Otherwise, we again define $\mathcal{M}_3$ similar to how $\mathcal{M}_2$ is defined, and show that the same result as in Lemma 10.4 holds for $\mathcal{M}_3$. We continue this procedure and define $\mathcal{V}_3, \mathcal{V}_4, \ldots$ until we reach an empty set $\mathcal{V}_l$, $l > 0$. This will certainly happen since $|\mathcal{V}_i| < |\mathcal{V}_{i-1}|$ for all $i \geq 1$ such that $\mathcal{V}_i \neq \emptyset$. Then the network has reached an equilibrium, which completes the proof of Theorem 10.1.

10.3.2 Biased-agents

We identify types of agents whose strategies are or become fixed after some amount of time, and provide a similar result to Theorem 10.1. Consider a strategy $r \in \{1, \ldots, m\}$. We call an agent

$i \in \mathcal{V}$ an *r-biased agent* if for any other strategy $q \in \{1, \dots, m\}, q \neq r$, and any neighbor $j \in \mathcal{N}_i$, the following holds

$$(\deg_i -1) \min_p \pi^i_{rp} + \pi^i_{rq} > (\deg_j -1) \max_p \pi^j_{qp} + \pi^j_{qr}$$

where $\deg_i$ denotes the degree of agent i. The definition implies that the payoff of an r-biased agent who plays r is more than all her neighbors playing some strategy different from r. This yields the following result, implying that r-biased agents' strategies remain invariant under imitation dynamics.

Lemma 10.5. *Given $r \in \{1, \dots, m\}$, if the strategy of an r-biased agent is r at some time, it does not change afterwards under the imitation update rule. Namely, for an r-biased agent i, it holds that*

$$(\exists k_1 : \ x_i(k_1) = r) \quad \Rightarrow \quad (x_i(k) = r \quad \forall k \geq k_1).$$

Proof. We prove by contradiction. Assume on the contrary there exists some time $k_2 > k_1$ at which agent i switches her strategy from r to $q \neq r$ that is the strategy of her neighbor, say agent j, with the maximum payoff at $k_2 - 1$. Then the strategy of agent i is r at $k_2 - 1$ and that of agent j is q. Hence, the payoff of agent i at $k_2 - 1$ satisfies

$$u_i(k_2 - 1) \geq (\deg_i -1) \min_p \pi^i_{rp} + \pi^i_{rq},$$

and that of agent j satisfies

$$u_j(k_2 - 1) \leq (\deg_j -1) \max_p \pi^j_{qp} + \pi^j_{qr}.$$

Now since agent i is r-biased, we conclude that $u_i(k_2 - 1) \geq u_j(k_2 - 1)$, which is in contradiction with the assumption that agent i changes her strategy to that of agent j at k_2. □

We say an agent is *biased* if the agent is r-biased for some $r \in \{1, \dots, m\}$, and *completely biased* if it is r-biased for all $r \in \{1, \dots, m\}$. When agents are completely biased, they become fixed from the beginning, regardless of their initial strategies.

Proposition 10.1. *Every network of completely biased agents is static under the asynchronous imitation update rule.*

The sufficient condition on biased agents in Proposition 10.1 can be relaxed when there are only two strategies available to the players. For convenience of notation, we refer to strategies 1 and 2 as A and B, respectively, when $m = 2$. So biased agents are either A-biased or B-biased. The following proposition states that unlike in the general case, the strategy of a biased agent becomes fixed when $m = 2$, regardless of its initial condition.

Proposition 10.2. *Every network of biased agents reaches an equilibrium under asynchronous imitation update rule, if $m = 2$.*

Proof. We prove by contradiction. Assume on the contrary that the strategy of a biased agent i never becomes fixed, implying that it keeps switching between A and B. This is in contradiction with Lemma 5 since if agent i is A-biased, her strategy never changes from A to B, and if she is B-biased, her strategy never switches from B to A. □

10.4 Convergence under arbitrary number of simultaneous updates

The result of Theorem 10.1 can be extended to when agents do not necessarily update asynchronously, i.e., when multiple agents can update simultaneously at any time step. Then the activation sequence becomes a sequence of sets $\mathcal{A}_k, k \in \{0, 1, \ldots\}$ consisting of agents that are active at k which may be 1, 2, up to all of the agents. Then the activation sequence becomes $\{\mathcal{A}_k\}_{k=0}^{\infty}$ where $A_k \subseteq \mathcal{V}$ consists of agents that are active at k, and where $|\mathcal{A}_i|$ and $|\mathcal{A}_j|$ are not necessarily equal for $i \neq j$. However, in order to guarantee convergence, the agents must satisfy a stronger condition than being opponent-coordinating; namely, their payoff matrices must satisfy

$$\pi^i_{p,p} + (\deg_i - 1)\pi^i_{p,p_{\min}} > \deg_i \pi^i_{p,p_{\max}} \tag{10.16}$$

where $\deg_i$ denotes the degree of agent i, $p_{\min}$ denotes the column of the minimum off-diagonal entry of the pth row in π^i and $p_{\max}$ denotes the column of the maximum off-diagonal entry of the pth row in π^i. We call agents satisfying the above condition *strongly opponent-coordinating* agents. Intuitively, each diagonal entry of such agents' payoff matrices is sufficiently greater then the off-diagonal entries in the same row. As with asynchronous updates, here we assume the activation sequence is persistent.

Theorem 10.2. *Every network of strongly opponent-coordinating agents reaches an equilibrium under the imitation update rule, regardless of how many agents update simultaneously at any time step.*

Proof. The proof follows the same steps as those in the proof of Theorem 1. Namely all the lemmas also hold for when the agents do not update asynchronously; however, now Case 2 in the proof of Lemma 10.1 requires a different argument, which we do in the following.

Case 2 in the proof of Lemma 10.1: A neighbor r of agent i, switches strategies at $k+1$. Agent r changes her strategy to that of one of her neighbors, say j, that has the highest payoff among the rest (j may equal i). This results in $j \in \mathcal{M}(k)$, implying that agent j does not switch strategies at $k+1$. So, the payoff of agent j at k equals

$$u_j(k) = \sum_{s \in \mathcal{N}_j} \pi^j_{x_j(k), x_s(k)},$$

and at $k+1$ equals

$$u_j(k+1) = \pi^j_{x_j(k), x_j(k)} + \sum_{s \in \mathcal{N}_j - \{r\}} \pi^j_{x_j(k), x_s(k+1)}.$$

Hence, according to the definition of strongly opponent-coordinating agents,

$$u_j(k+1) - u_j(k) > 0. \tag{10.17}$$

The remaining can be done the same as in that of Lemma 10.1. This modified argument also applied to similar other lemmas such as Lemma 10.3, leading to the proof of the theorem. □

For the especial case of $m = 2$, a opponent-coordinating agent turns out to be also strongly row-coordinating, yielding the following result.

Corollary 10.1. *Every network of opponent-coordinating agents with only two available strategies, i.e., $m = 2$, reaches an equilibrium under the imitation update rule, regardless of how many agents update at the same time.*

Proof. When $m = 2$, an opponent-coordinating agent is also strongly opponent-coordinating, i.e., Condition (8) in the main text is also in force for such agents. This is because $\pi^i_{p,p_{\max}} = \pi^i_{p,p_{\min}} < \pi^i_{p,p}$ for every opponent-coordinating agent i when $m = 2$. The proof then follows readily from Theorem 10.2. □

It is worth mentioning that all convergence results provided so far require the persistence assumption, yet based on the proofs, the assumption is only needed to guarantee that the final state is an equilibrium.

Remark 10.1. *Theorems 10.1 and 10.2 hold even when the persistence assumption does not hold; however, then after some finite time, the network reaches and remains at a final state that may not be an equilibrium.*

10.5 Non-convergence behavior

In this section, we provide some counterexamples to demonstrate cases in which networks containing non-opponent-coordinating agents may never reach equilibrium.

10.5.1 Three anti-coordinating agents in a line

In a network containing only two agents, any asynchronous imitation will result in an equilibrium. Therefore, the smallest network in which asynchronous imitation may lead to non-convergence is one consisting of three agents. Such a network can be constructed out of three anti-coordinating agents connected in a path, *i.e.* one edge connects agents 1 and 2 and another edge connects agents 2 and 3 (see Fig. 10.1). The minimum number of strategies required for non-convergence

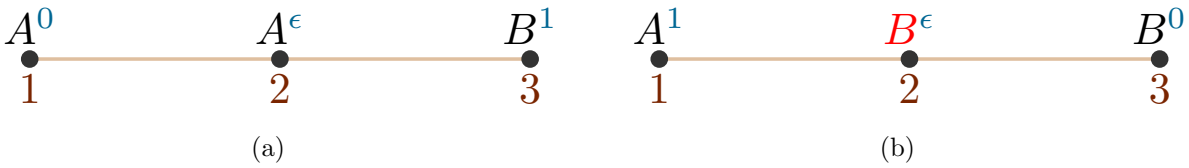


Figure 10.1: **Three anti-coordinating agents in a line.** The payoff matrix of agents 1, 2 and 3 are π_0, $\epsilon\pi_0$ and π_0, respectively, where π_0 is defined in (10.18). The superscripts in blue indicate the payoffs. (a) Agent's initial strategies. (b) Agent's strategies at the first time step an agent switches strategies. The agent's strategies at the next time step that an agent switches will be the same as part (a), resulting in a non-convergence behavior.

is two, i.e., $m = 2$. We refer to strategies 1 and 2 as A and B, respectively. Define the following payoff matrix

$$\pi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{10.18}$$

and let $\pi^1 = \pi^3 := \pi_0$, and $\pi^2 = \epsilon\pi_0$, where $\epsilon < 1$. Suppose the initial strategies of this network are $x(0) := (A, A, B)$ yielding payoffs $(0, \epsilon, 1)$. Neither agent 1 nor 3 will change strategies, but if agent 2 is activated at time 1, then $x(1) := (A, B, B)$, yielding the new payoffs $(1, \epsilon, 0)$. From this state, again neither agent 1 nor 3 will switch and if agent 2 is activated again at time 2, then $x(2) := (A, A, B) = x(0)$, resulting in a cycle of length 2, and the network will never reach equilibrium. This configuration can also appear embedded in much larger networks and demonstrates that imitative anti-coordinating agents who receive less payoff than their neighbors for the same types of interactions can quite easily be made to waver between the strategies of more steadfast neighbors.

10.5.2 Extension to a homogeneous network

We extend the previous example to a network of homogeneous agents with the payoff matrix

$$\pi^i = \begin{pmatrix} R & S \\ T & P \end{pmatrix}, \quad P > 0, \quad 0 < R < T, \quad S > P + R. \tag{10.19}$$

For this, we need to add α_1 initially A-playing neighbors to agent 1, and α_2 initially B-playing neighbors to agent 3 (see Fig. 10.2) where $\alpha_1, \alpha_2 \geq 1$ are positive integers satisfying

$$R\alpha_1 \geq S, \tag{10.20}$$
$$P\alpha_2 \geq T, \tag{10.21}$$
$$\lfloor \frac{R}{T}(\alpha_1 + 1) - \frac{P}{T}\alpha_2 \rfloor = 0. \tag{10.22}$$

Such α_1 and α_2 exist since one can first choose them to be large enough to satisfy (10.20) and (10.21), then increase α_2 so that $R(\alpha_1 + 1) - P\alpha_2 < 0$, and then start increasing α_1 until the first time $R(\alpha_1+1)-P\alpha_2$ becomes positive. Since $R < T$, it then follows that $0 < R(\alpha_1+1)-P\alpha_2 < T$, satisfying (10.22), proving the existence of α_1 and α_2.

As in the previous example, we assume that the initial strategies of agents $1, 2$ and 3 are (A, A, B). Sharing the same strategies as those of their neighbors, the adjoining agents neither switch their own strategies at $k = 1$, nor cause agents 1 and 3 to do so. The payoffs of agents 1, 2 and 3 at $k = 0$ are respectively

$$u_1 = (\alpha_1 + 1)R, \ u_2 = R + S, \ u_3 = \alpha_2 P + T. \tag{10.23}$$

Therefore, $u_2 < u_1$ according to (10.20). Moreover, in view of (10.22),

$$\frac{R}{T}(\alpha_1 + 1) - \frac{P}{T}\alpha_2 < 1 \overset{T>0}{\Longrightarrow} u_1 < u_3.$$

Hence, $u_2 < u_1 < u_3$. So only the strategy of agent 2 can change. Let this take place at some time $k_1 \geq 1$. Then the strategies of agents $1, 2$ and 3 at k_1 will be (A, B, B). Correspondingly the agents' new payoffs will be

$$u_1 = \alpha_1 R + S, \ u_2 = T + P, \ u_3 = (\alpha_2 + 1)P.$$

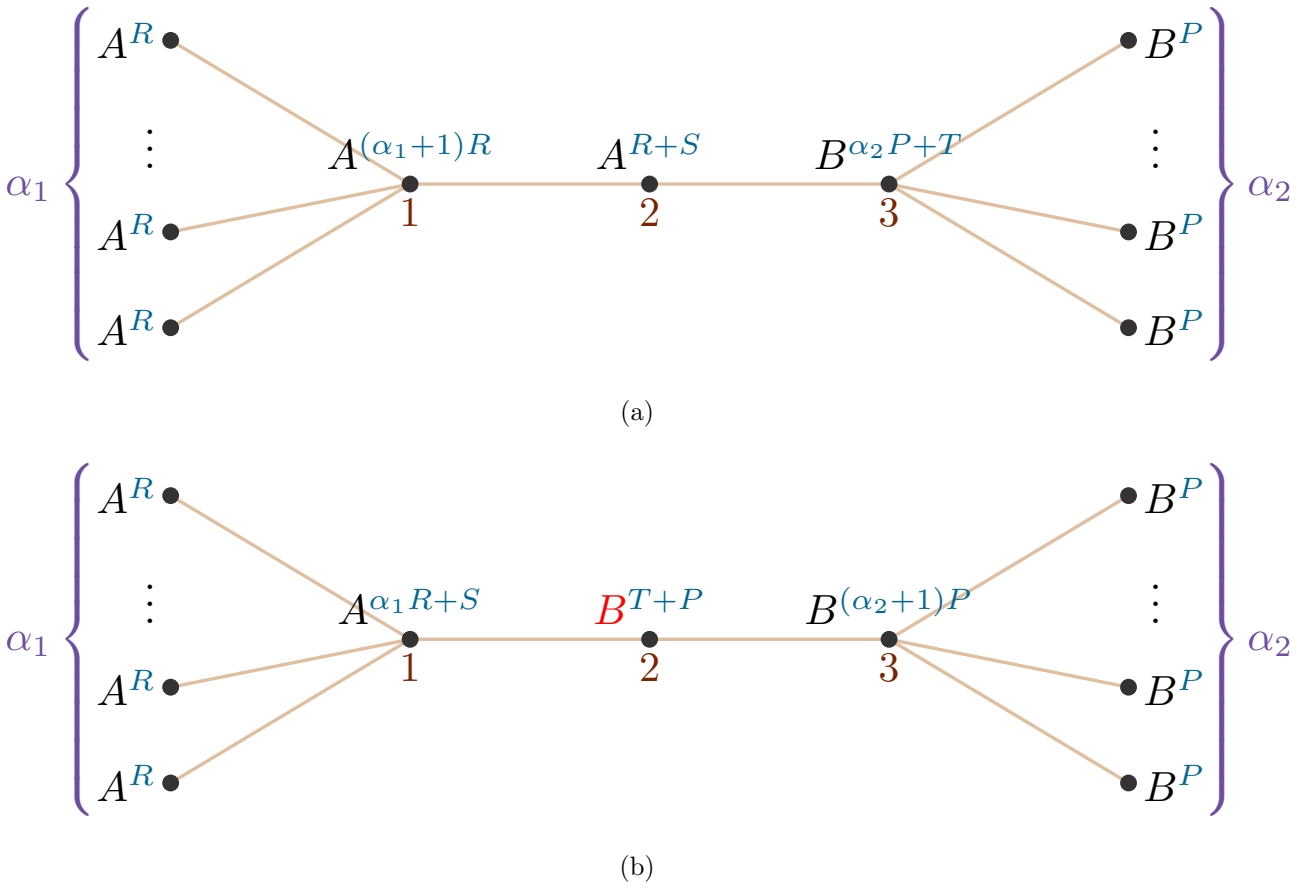


Figure 10.2: **Extension of the example in Section 10.5.1 to a homogeneous network.** All agents' payoff matrices equal that in (10.19), implying a homogeneous network. The superscripts in blue indicate the payoffs. (a) Agent's initial strategies. (b) Agent's strategies at the first time step an agent switches strategies. The agent's strategies at the next time step that an agent switches will be the same as part (a), resulting in a non-convergence behavior.

Therefore, $u_2 < u_3$ according to (10.21). On the other hand, from (10.19), $S > P + R$ and $T > 0$. So, in view of (10.22),

$$\frac{R}{T}(\alpha_1 + 1) - \frac{P}{T}\alpha_2 + \frac{S - P - R}{T} > 0 \overset{T>0}{\Longrightarrow} u_3 < u_1.$$

Hence, $u_2 < u_3 < u_1$. So again only agent 2's strategy can change. Therefore, the next time the network reaches a new state, it will revert back to its state at $k = 0$, resulting in a non-converging long-run behavior. The same can be shown when the last three inequalities in (10.19) are replaced with $R > 0$, $0 < P < S$ and $T > P + R$. Both cases correspond to significant classes of payoff matrices, implying that even networks of homogeneous agents may often not converge. This is further supported by the next example.

10.5.3 Ring of asynchronous, homogeneous, non-opponent-coordinating agents

Consider a ring network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{E} = \{(1,2),(2,3),\dots,(n-1,n),(n,1)\}$. The network is homogeneous in that the agents' payoff matrices $\pi^i, i \in \mathcal{V}$, are the same and equal to

$$\pi^i = \begin{pmatrix} R & S \\ T & P \end{pmatrix}, \qquad R < T,\ T + P < R + S < 2T. \tag{10.24}$$

The agents' initial strategies are as follows: $x_1(0) = B$ and $x_i(0) = A$ for all $i \in \mathcal{V} - \{1\}$ (see Fig. 10.3(a)). We show that the network fluctuates between two classes of states: first,

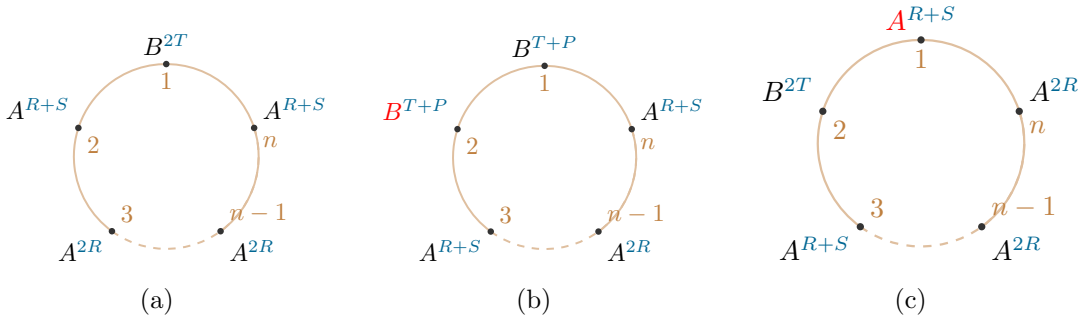


Figure 10.3: **Ring of asynchronous homogeneous non-opponent-coordinating agents.** All agents' payoff matrices equal that in (10.24), implying a homogeneous network. The superscripts in blue indicate the payoffs. (a) Agent's initial strategies. It holds that $x(0) \in \hat{\mathcal{X}}$. (b) Agent's strategies at $k = 1$ when agent 2 was active at $k = 0$. It holds that $x(1) \in \bar{\mathcal{X}}$. (c) Agent's strategies at $k = 2$ when agent 1 was active at $k = 1$. It holds that $x(1) \in \hat{\mathcal{X}}$, meaning that the strategy configuration has reached the same structure as it had at $k = 0$. A similar process takes place for any other activation sequence. Therefore, the network game never reaches an equilibrium.

$\hat{\mathcal{X}} = \{\hat{x}^i \,|\, i \in \mathcal{V}\}$, where $\hat{x}^i \in \{A, B\}^n$ and is defined by

$$\hat{x}^i_j = \begin{cases} B & j = i \\ A & j \neq i \end{cases},$$

and second, $\bar{\mathcal{X}} = \{\bar{x}^i \,|\, i \in \mathcal{V}\}$ where $\bar{x}^i \in \{A, B\}^n$ and is defined by

$$\bar{x}^i_j = \begin{cases} B & j = i \text{ or } j = i + 1 \\ A & \text{otherwise} \end{cases}$$

where $i + 1$ is counted modulo n. Note that $x(0) = \hat{x}^1$ under this definition. First, we look at when the network game starts at a state in $\hat{\mathcal{X}}$, say $\hat{x}^i, i \in \mathcal{V}$. The payoff of agent i is then $2T$, that of her two neighbors, i.e., $i - 1$ and $i + 1$, is $R + S$ and that of the rest are $2R$. Hence, in view of (10.24), if any of the agents $i - 1$ and $i + 1$ are active, they will switch their strategies to B. However, agent i will not switch strategies, and neither will any of the other agents since both they and their neighbors play A. So at the next time step, the network either stays at $\hat{x}^i$ or moves

to one of the two states $\bar{x}^i$ or $\bar{x}^{i-1}$ in the second class (see Fig. 10.3(b)). Now we look at when the network game starts at a state in $\hat{\mathcal{X}}$, say $\bar{x}^i, i \in \mathcal{V}$. Then the following holds for the agents' payoffs

$$u_j = \begin{cases} T+P & j = i \text{ or } j = i+1 \\ R+S & j = i-1 \text{ or } j = i+2 \\ 2R & \text{otherwise} \end{cases}.$$

Hence, in view of (10.24), if any of the agents i and $i+1$ are active, they will switch their strategies to A. The rest of the agents will not switch. So at the next time step, the network either stays at $\bar{x}^i$ or moves to one of the states $\hat{x}^i$ or $\hat{x}^{i+1}$ in the first class (see Fig. 10.3(c)). Now if in addition, the activation sequence is persistent, the network cannot remain in any of the classes forever, resulting in a fluctuation between the two, a non-convergence behavior.

One can further construct an activation sequence that results in a cycle, for example, $\mathcal{S} = \{i^0, i^1, \ldots\}$ where

$$i^k = \begin{cases} \frac{k}{2}+2 & k = 0, 2, 4, \ldots \\ \frac{k-1}{2}+1 & k = 1, 3, 5, \ldots \end{cases},$$

resulting in $\mathcal{S} = \{2, 1, 3, 2, 4, 3, \ldots\}$. Then the network evolves as $\hat{x}^1, \bar{x}^1, \hat{x}^2, \bar{x}^2, \ldots, \hat{x}^n, \bar{x}^n, \hat{x}^1, \bar{x}^1, \ldots$, namely

$$x(k) = \begin{cases} \hat{x}^{\frac{r_1}{2}+1} & k = 0, 2, 4, \ldots \\ \bar{x}^{\frac{r_2+1}{2}} & k = 1, 3, 5, \ldots \end{cases}$$

where r_1 and r_2 return the remainders of the devisions of k by $2n-2$ and $2n-1$, respectively. So the system undergoes a cycle of length $2n$. Note that, however, the probability of such an activation sequence is zero.

Such behavior never shows up for a network of opponent-coordinating agents in view of Theorem 10.1. Indeed for an opponent-coordinating agent, we have that

$$P > T \Rightarrow T + P > 2T,$$

violating the inequality in (10.24). Other types, however, may satisfy the inequality, including most well-known types of agents. For example, the payoff matrices

$$\pi^{SD_1} = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \pi^{SD_2} = \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$$

corresponding to two snowdrift games, both satisfy (10.24). So any homogeneous network of agents with the payoff matrix π^{SD_1} or homogeneous network of agents with the payoff matrix π^{SD_2} never converges to a single state.

The results can further be extended to heterogeneous networks, by modifying the condition in (10.24) as follows. Assume that each agent $i \in \mathcal{V}$ has a possibly unique payoff matrix as

$$\pi^i = \begin{pmatrix} R_i & S_i \\ T_i & P_i \end{pmatrix}, \qquad R_i, S_i, T_i, P_i \in \mathbb{R},$$

and that for any (not necessarily distinct) $i, j \in \mathcal{V}$, it holds that

$$R_i < T_j \quad \text{and} \quad T_i + P_i < R_j + S_j \quad \text{and} \quad R_j + S_j < 2T_i. \tag{10.25}$$

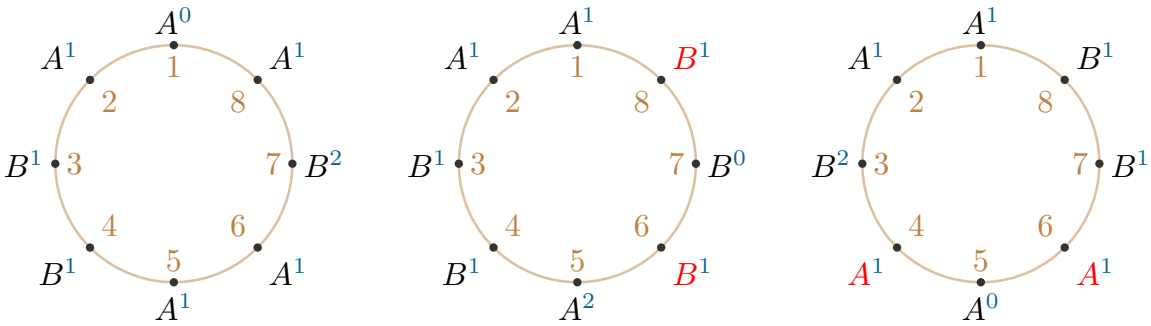
Then it can be verified that again starting from $\hat{x}^1$, the network does not converge to a single state. As a result, any network of agents, some of which having π^{SD_1} and others π^{SD_2}, never converges, since then (10.25) is in force.

10.5.4 Long cycles in synchronous networks

When networks of anti-coordinating agents update in full synchrony under imitation dynamics, it is possible for relatively long cycles to emerge. This contrasts with synchronous best-response dynamics, in which it has been shown that cycles of length at most 2 can occur [2]. Following is an example of how cycles of length $\frac{n}{2}$ can appear in rings of synchronous anti-coordinating agents. Consider a ring of $4p$ agents, where $p \in \{2, 3, \dots\}$, and let the payoffs of each agent be given by the anti-coordinating matrix π_0 defined in (10.18). Suppose that the initial strategies are as follows:

$$x_i(0) = \begin{cases} A & \text{if } (i-1) \bmod 4 \leq 1 \text{ or } i = n \\ B & \text{otherwise} \end{cases}.$$

For example, in a ring of 8 agents, the initial strategies are $(A^0, A^1, B^1, B^1, A^1, A^1, B^2, A^1)$, where the superscripts indicate the initial payoffs (see Fig. 10.4). At the next time step, both neighbors



(a) (b) (c)

Figure 10.4: **Ring of synchronous homogeneous non-opponent-coordinating agents.** All agents' payoff matrices equal that in (10.18), implying a homogeneous network. The superscripts in blue indicate the payoffs. (a) Agent's initial strategies. (b) Agent's strategies at $k = 1$. (c) Agent's strategies at $k = 2$. The resulting strategy configuration is the same as that at $k = 0$ but shifted by 4 agents. Therefore, the network game undergoes a cycle of length 4 and never converges to an equilibrium.

of the B-agent whose payoff is 2 will switch to A and all other agents will keep their strategies, resulting in the configuration $(A^1, A^1, B^1, B^1, A^2, B^1, B^0, B^1)$ at time 1. At the next time step, we will have $(A^1, A^1, B^2, A^1, A^0, A^1, B^1, B^1)$, which is the same as the initial configuration shifted by 4 agents. It is straightforward to see that after two more time steps, the network will return to the initial strategy state, completing a cycle of length 4. Indeed for rings of n agents as described above, a cycle of length $\frac{n}{2}$ persists.

10.6 Concluding remarks

We have shown that any network of opponent-coordinating agents who asynchronously imitate the strategy of their most successful neighbor will reach an equilibrium state. This finds particular interest in social contexts where agents' payoffs when playing a particular strategy are highest when their neighbors also play that strategy, e.g., financial investments, the spread of social norms, technological innovations and voting opinions. Our results imply that regardless of how individuals are linked (network topology) and how differently they perceive the interaction outcomes (heterogeneity of the payoff matrices), the network reaches an equilibrium state, in which all agents are playing the highest-earning strategies in their respective neighborhoods. If the agents are in addition strongly opponent-coordinating, i.e., agents' payoffs when playing a particular strategy are sufficiently higher when their neighbors play the same strategy as they do, then the network still converges to an equilibrium even when the individual updates are partially or fully synchronous. Consequently, non-convergence behavior in such situations may imply the presence of individuals perceiving the social context extremely different from what it normally is, e.g., anti-coordinating agents who earn more when they play the opposite strategy of the majority of their neighbors. Another possibility is the existence of individuals who update not based on the success of their neighbors but on the frequency of their neighbors' strategies (best-response update rule). The convergence results also allow for investigation of convergence time, stationary distribution of the strategies and methods to control the final state.

The second and more important contribution of this chapter is however to show that convergence under imitation is a rare phenomenon. We have shown this by establishing the potential for non-convergence of networks containing non-opponent-coordinating individuals. We have provided networks of heterogeneous agents that never converge to an equilibrium state, the simplest of which consists of three agents connected on a line, in which the outer agents are biased while the middle agent is more easily swayed. Such simple configurations may show up as part of any larger heterogeneous network, premising that convergence under the imitation update rule is an uncommon phenomenon, especially in large networks. We have provided further evidence for this hypothesis by extending the example to significant classes of homogeneous payoff matrices, and providing another homogeneous example on ring networks, both leading to non-converging behavior. The lower expectation of convergence by imitation is further highlighted if we compare these observations with what the best-response update rule leads to in the case of two available strategies: any homogeneous population and any heterogeneous population of coordinating agents and heterogeneous population of anti-coordinating agents reaches an equilibrium under the best-response. The comparison also makes frequency-based learners (who update based on best-response update rule) more promising in leading the whole network to satisfactory decisions than success-based learners (who update based on the imitation update rule) that are less independent (or less self-confident) and ignore their own situation. This can have consequences in media policies for stabilizing networks exhibiting chaotic fluctuations such as stock markets by inducing companies to make independent decisions, rather than mimicking top firms.

Chapter 11

Control of asynchronous imitation dynamics on networks

Imitation is prevalent in many populations of decision-making agents. Using our recent convergence results for asynchronous imitation dynamics on networks, we consider how such networks can be efficiently driven to a desired equilibrium state by offering payoff incentives or rewards for using a certain strategy, either uniformly or targeted to individuals. In particular, if agents playing a given strategy receive maximum payoff when their neighbors play the same strategy, then we show that providing incentives to any network in an equilibrium state will result in convergence to a *unique* equilibrium. This result allows the computation of optimal uniform incentives using a binary search algorithm. When different incentives can be offered to each agent, we propose an algorithm to select which agents should be targeted based on iteratively maximizing a weighted ratio of the number of agents who adopt the desired strategy to the payoff incentive required to get those agents to switch. Simulations demonstrate that the proposed algorithm computes near-optimal targeted payoff incentives for a range of networks and payoff distributions in coordination games.

11.1 Introduction

Networks in which agents make decisions by imitating their most successful neighbors appear frequently in sociology, biology, economics, and engineering [171, 176]. Such networks of *success-based* learners often exhibit complex non-convergent behaviors even when the agents are homogeneous. In other words, focusing on the success of others hinders the agents from reaching satisfactory decisions, as discussed in Chapter 10. This non-convergence relates to volatility and instability of networks which can have consequences ranging from costly inefficiencies to catastrophic failures. Imitation is also known to lead to selfish behaviors in various social contexts [176], which can manifest as social dilemmas such as tragedy of the commons, in which the pursuit of selfish goals leads to globally suboptimal outcomes. However, in many of these cases it may be possible to circumvent the undesired global outcomes by administering some small control input to the agents, locally. Given that this could require a large amount of total control effort, it is critical to develop methods for achieving these goals as efficiently as possible. Game theory is widely

used to model distributed optimization and learning in large populations of autonomous agents [39, 98, 90, 182, 155, 58, 51, 113], but more specifically, *evolutionary game theory* allows for strategies to propagate through populations by means other than rational choice, and therefore provides an ideal framework to model networks of imitative agents [33, 115, 189, 138, 139].

Researchers have investigated how to drive populations to consensus in a desired strategy by offering payoff incentives or economical benefits to the individuals [109, 189, 144, 136]. Several approaches have been used to control such networks, three of which we have formulated in Chapter 9. First is the *uniform reward control* where a central regulating agency is assumed to be able to uniformly change the payoffs of every agent so that they play the desired strategy, and the goal is to do this by offering the minimum reward. Examples include equilibrium shifting in stochastic snowdrift games by means of payoff modifications [91]. Second is *targeted reward control* where the regulating agency has now the power to target individual agents and offer them independent sufficient rewards to lead the network more efficiently to the state where every individual plays the desired strategy [144, 186]. Third is *budgeted targeted reward control* which is when the incentive budget is limited, and the goal is to maximize the number of individuals playing the desired strategy subject to the budget constraint. In addition, there may be situations where we can directly control the strategies of the agents [145, 144]. The goal then is to find the minimum number of agents required to adopt a desired strategy, so that the rest of the agents in the network will eventually follow the same strategy. We call this control problem, *direct strategy control*. There are two key properties that facilitate the design of control algorithms for all the above cases; if the network is at some equilibrium, then providing incentives to the agents should (i) cause no agent to switch away from a desired strategy, and (ii) result in convergence of the network to a unique equilibrium state. In Chapter 9, we have demonstrated these properties for coordinating agents, who earn more if they play the same strategy as that of the majority of their neighbors, under the asynchronous best-response update rule, and designed near-optimal algorithms for the first three control problems, which also work well for anti-coordinating agents. However, it remains to be discovered the conditions under which networks of imitative agents can be driven to a desired equilibrium.

In this chapter, we find efficient incentive-based control algorithms for three different control problems on finite networks of heterogeneous decision-making individuals who asynchronously imitate their highest earning neighbors. We start by building a general framework for asynchronous network games with two available strategies, A and B. Our main theoretical contribution is to show that in any such network game, regardless of the update rule, if all agents are *A-coordinating*, i.e., agents who update to strategy A would also do so if they had more neighbors playing A, then providing incentives to the agents when the network is at equilibrium (i) causes no agent to switch from A to B, and (ii) leads the network to a unique equilibrium regardless of the agents' activation sequence. Next we prove that networks governed by imitation dynamics satisfy these conditions provided that all agents are *opponent coordinating*, i.e., agents' payoffs are maximized when their neighbors play the same strategy that they do. These results make possible the design of efficient control algorithms using payoff incentives, to guarantee the convergence of networks of imitating agents to a desired strategy. In particular, we propose the *Iterative Potential-to-Reward Optimization (IPRO)* algorithm, inspired by the similar approach for controlling best-response networks in Chapter 9, that selects which agents should be targeted based on iteratively maximizing a weighted ratio of the number of agents who adopt the desired strategy to the payoff incentive

required to get those agents to switch. Simulations show that the IPRO algorithm performs near-optimal in a variety of cases and outperforms other incentive targeting algorithms based on highest degree or maximum earnings.

11.2 Asynchronous network games

Although the primary focus of this chapter is imitation, some of the results that appear later on apply to a broader class of dynamics, so we present in this section a generalized framework of two-strategy asynchronous games on networks.

Consider an undirected network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The nodes $\mathcal{V} = \{1, \dots, n\}$ represent agents who, at each time $k \in \{0, 1, \dots\}$, play 2-player games with all of their neighbors, with whom they share an edge in the set $\mathcal{E}$. Every agent starts with one of the strategies A or B at $k = 0$. Then, at each time step, the agents accumulate payoffs from all neighbors. The possible payoffs of an agent $i \in \mathcal{V}$ against another agent j are summarized in the *payoff matrix* $\pi^i \in \mathbb{R}^{2\times 2}$ whose entry π_{x_i,x_j} represents agent i's payoff when playing strategy p against a neighbor playing strategy q where $p, q \in \{A, B\}$, and $A = 1$, $B = 2$ for the purposes of matrix indexing. We stack all payoff matrices in a 3-dimensional matrix $\pi \in (\mathbb{R}^{2\times 2})^n$. The payoff of agent i against all neighbors at time k is given by

$$u_i(k) = \sum_{j \in \mathcal{N}_i} \pi^i_{x_i(k),x_j(k)}$$

where $\mathcal{N}_i$ is the set of agent i's neighbors. After collecting all payoffs, one random agent i is activated at time k and updates to a new strategy at time $k + 1$ according to some update rule, which we denote by $\mathfrak{R}$:

$$x_i(k+1) = \begin{cases} A & \text{if } f_i(x(k)) = \{A\} \\ B & \text{if } f_i(x(k)) = \{B\} \\ z_i & \text{if } f_i(x(k)) = \{A, B\} \end{cases} \tag{11.1}$$

where $f_i : \{A, B\}^n \to \{A, B, \{A, B\}\}$. z_i is fixed and equals either A, B or $x_i(k)$. By a *network game* $\Gamma : (\mathcal{G}, \pi, \mathfrak{R})$ we mean a network $\mathcal{G}$ of agents with payoff matrices π, who update based on $\mathfrak{R}$. We do not prescribe any particular process for driving the activation sequence, but we do assume that every agent is activated infinitely many times as time goes to infinity.

Agent's strategies evolve under the update rule according to the sequence in which agents activate, and may converge to an equilibrium state or continue to fluctuate. An equilibrium of the dynamics is a state x^* at which none of the agents violate the update rule, implying that if $x(k) = x^*$ for some $k \geq 0$, then $x(k + 1) = x^*$, regardless of which agent is active at time k. Our goal is to control the dynamics of network games with the imitation update rule, by offering payoff incentives, to reach or get as close as possible to a desired equilibrium state where every agent plays strategy A. By offering payoff incentives to a network, we mean offering a (possibly unique) non-negative reward to every agent in the network for playing strategy A. The visualization of this in the payoff matrix, is to add non-negative constants to the entries of the row corresponding to strategy A. We investigate when a network game with any update rule reaches a unique equilibrium after offering payoff incentives, leading to the design of efficient control algorithms.

11.3 Unique equilibrium convergence of A-coordinating network games

Equilibrium convergence is a key property of network games, and can be guaranteed for certain classes of update rules and agent payoff matrices as discussed in Chapters 8 and 10. However, it is not generally the case that such networks will converge to a *unique* equilibrium, a property which is highly desirable for the design of efficient and predictable control algorithms. Here we establish conditions on the agents and update rule under which unique equilibrium convergence can be guaranteed.

We say a network game is A-*coordinating* if any agent who updates to strategy A would also do so if some agents currently playing B were instead playing A. Formally, we have the following definition.

Definition 11.1. *We say a network game $(\mathcal{G}, \pi, \mathfrak{R})$ is A-coordinating if for any two strategy vectors $y, z \in \{A, B\}^n$ satisfying*

$$y_i = A \Rightarrow z_i = A \qquad \forall i \in \mathcal{V}, \tag{11.2}$$

the following holds

$$f_i(y) = \{A\} \Rightarrow f_i(z) = \{A\} \qquad i \in \mathcal{V} \tag{11.3}$$

and

$$f_i(y) = \{A, B\} \Rightarrow A \in f_i(z) \qquad i \in \mathcal{V}. \tag{11.4}$$

The A-coordinating property implies that having more A-playing agents in the network may lead agents to switch from B to A and preserves those already playing A, yielding a monotone behavior in agents' strategies. We say a network game is A-*monotone* if after offering payoff incentives to the network when it is at any equilibrium, no agent ever switches from A to B.

Proposition 11.1. *Every A-coordinating network game is A-monotone.*

We need the following lemma for the proof.

Lemma 11.1. *Consider an A-coordinating network game $(\mathcal{G}, \pi, \mathfrak{R})$. If for some agent $i \in \mathcal{V}$, one of the following holds at some time $k \geq 0$:*

1. *$A \in f_i(x(k))$ and $A \notin f_i(x(k+1))$,*
2. *$B \notin f_i(x(k))$ and $B \in f_i(x(k+1))$,*

then an agent has switched from A to B at time $k+1$.

Proof. We prove by contradiction. Assume the negation of Lemma 11.1 holds for a network at some time k and let $x(k)$ denote the state of the network at that time. Since no agent has switched from A to B at time $k+1$, the vectors $y = x(k)$ and $z = x(k+1)$ satisfy Condition (11.2). Now if Case 1 takes place, then either $f_i(y) = \{A\}$, violating (11.3) or $f_i(y) = \{A, B\}$, violating (11.4), a contradiction, yielding the result. If on the other hand, Case 2 takes place, then $f_i(y) = \{A\}$, violating (11.3) since $B \in f_i(z)$, a contradiction, leading to the proof. □

Proof of Proposition 11.1. We again prove by contradiction. Assume the contrary and let $k_1 \geq 1$ be the first time that some agent i switches from A to B. Then one of the following cases holds:

Case 1: $A \notin f_i(k_1 - 1)$. On the other hand, either the strategy of agent i is A at 0, yielding $A \in f_i(0)$ since the network is at equilibrium at 0, or there is some time $\hat{k} \in [0, k_1 - 2]$ such that agent i switches to A at $\hat{k}+1$, yielding $A \in f_i(\hat{k})$. So in any case, there exists some $k_0 \in [0, k_1 - 1]$ such that $A \in f_i(k_0)$. Therefore, since $A \notin f_i(k_1 - 1)$ there exists some time $k_2 \in [k_0, k_1 - 2]$ such that $A \in f_i(k_2)$ and $A \notin f_i(k_2+1)$. In view of Lemma 11.1, this implies that an agent has switched from A to B at $k_2 + 1$, a contradiction since $k_1 > k_2 + 1$ is the first time that such a switch takes place, yielding the result.

Case 2: $f_i(k_1 - 1) = \{A, B\}$ and $z_i = B$. On the other hand, either the strategy of agent i is A at 0, yielding $f_i(0) = \{A\}$ since $z_i = B$ and that the network is at equilibrium at 0, or there is some time $\hat{k} \in [0, k_1 - 2]$ such that agent i switches to A at $\hat{k}+1$, yielding $f_i(\hat{k}) = \{A\}$. So in any case, there exists some $k_0 \in [0, k_1 - 1]$ such that $f_i(k_0) = \{A\}$. Therefore, since $f_i(k_1 - 1) = \{A, B\}$ there exists some time $k_2 \in [k_0, k_1 - 2]$ such that $B \notin f_i(k_2)$ and $B \in f_i(k_2 + 1)$. In view of Lemma 11.1, this implies that an agent has switched from A to B at $k_2 + 1$, a contradiction, leading to the proof. □

Moreover, we say that a network *switches sequence-independently* if, after offering incentives to one or more agents when the network is at equilibrium, any agent who switches from B to A under one activation sequence will do so under any activation sequence (possibly at a different time).

Proposition 11.2. *Every A-coordinating network switches sequence-independently.*

This can be explained intuitively as follows. Consider two activation sequences $\mathcal{S}^1 := \{i^0, i^1, \ldots\}$ and $\mathcal{S}^2 := \{j^0, j^1, \ldots\}$. Let i be the first agent who switches from B to A under S^1 (we know from Proposition 11.1 that switches from A to B are impossible). Agent i will also switch from B to A at the first time that she is active under S^2 since up to that time, agents may have switched only from B to A under S^2, which is "in favor" of other agents updating to A, due to the A-coordinating nature of the network game. Then by induction, the same can be shown for the second and later agents who switch their strategies from B to A under S^1. We formalize and prove this statement in the following Lemma, borrowing some ideas from our previous result in Chapter 9. Let t_0 be the first time when agent j^0 is active in $\mathcal{S}^1$. Then for $s = 1, 2, \ldots$, define k_s as the first time after k_{s-1} that agent j^s is active in $\mathcal{S}^1$. The time k_s exists because of the assumption that each agent is activated infinitely many times. Denote by x_i^1 and x_i^2, the strategies of agent i under the activation sequences $\mathcal{S}^1$ and $x_i^2(t)$, respectively.

Lemma 11.2. *Consider an A-coordinating network game* $(\mathcal{G}, \pi, \mathfrak{R})$ *which is at equilibrium at time* 0. *Suppose that some payoff incentives are offered at time* 0. *Then given any two activation sequences* $\mathcal{S}^1 = \{i^0, i^1, \ldots\}$ *and* $\mathcal{S}^2 = \{j^0, j^1, \ldots\}$, *the following holds for* $s = 0, 1, 2 \ldots$

$$x_{j^s}^2(s+1) = A \quad \Rightarrow \quad x_{j^s}^1(k_s + 1) = A. \tag{11.5}$$

Proof. We prove by induction on s. The statement is first shown for $s = 0$. Suppose $x_{j^0}^2(1) = A$. The initial strategy of agent j^0 is the same under both sequences, i.e., $x_{j^0}^1(0) = x_{j^0}^2(0)$. Therefore, since the network game is A-monotone in view of Proposition 11.1, no agent has switched to B

before time k_0, under $\mathcal{S}^2$. So since the network game is A-coordinating, it follows that $x^1_{j^0}(k_0+1) = A$ if $x^1_{j^0}(1) = A$, verifying (11.5) for $s = 0$.

Now assume that (11.5) holds for $s = 0, 1, \ldots, r-1$. Suppose $x^2_{j^r}(r+1) = A$. Now since (11.5) holds for all $s = 0, 1, \ldots, r-1$, and because of Proposition 11.1, we obtain that if any agents have switched and hence fixed their strategies from B to A under $\mathcal{S}^2$ up to the time $k = r$, they have also done so under $\mathcal{S}^1$ up to any time $k \geq k_{r-1} + 1$. Moreover, no agent has switched from B to A under $\mathcal{S}^1$. Thus, the strategy vectors $y = x^2(r)$ and $z = x^1(k_r)$ satisfy the condition in (11.3). Hence, (11.5) is true for $s = r$ since the network game is A-coordinating. □

Proof of Proposition 11.2. The proof follows directly from Lemma 11.2. □

These two properties of A-coordinating network games lead to the main result of this section. We say that a network game is *uniquely convergent*, if after offering some payoff incentives when the network is at equilibrium, the network will again reach an equilibrium state which is unique and does not depend on the sequence in which agents activate.

Theorem 11.1. *Every A-coordinating network game is uniquely convergent.*

Proof. According to Proposition 11.1, no agent switches from B to A. Since every agent is activated infinitely many times, it follows that the network reaches an equilibrium state in finite time. It remains to prove the uniqueness of the equilibrium for all activation sequences, which we do by contradiction. Assume that there exist two activation sequences $\mathcal{S}^1 = \{i^0, i^1, \ldots\}$ and $\mathcal{S}^2 = \{j^0, j^1, \ldots\}$ that drive the network to two distinct equilibrium states, implying the existence of an agent q whose strategy is different at the two equilibria, say B under the equilibrium of $\mathcal{S}^1$ and A under the equilibrium of $\mathcal{S}^2$. However, in view of Proposition 11.2, agent q's strategy will become A at some time under $\mathcal{S}^1$, and will not change afterwards because of Proposition 11.1, a contradiction, completing the proof. □

11.4 Imitation update rule

The imitation update rule $\mathcal{I}$ dictates that agent i, active at time k, updates at time $k+1$ to the strategy of the agent earning the highest payoff at time k in the neighborhood $\mathcal{N}_i \cup \{i\}$. If several agents with different strategies earn the highest payoff, we assume agent i does not switch, namely

$$x_i(k+1) = \begin{cases} A & \text{if } \mathcal{S}_i^M(k) = \{A\} \\ B & \text{if } \mathcal{S}_i^M(k) = \{B\} \\ x_i(k) & \text{if } \mathcal{S}_i^M(k) = \{A, B\} \end{cases} \tag{11.6}$$

where $\mathcal{S}_i^M(k)$ is the set of strategies resulting in the maximum payoff at time k in the neighborhood of agent i, that is

$$\mathcal{S}_i^M(k) \triangleq \left\{ x_j(k) \,\middle|\, u_j(k) = \max_{r \in \mathcal{N}_i \cup \{i\}} u_r(k) \right\}.$$

By comparing (11.6) with (11.1), we obtain that $f_i = S_i^M$ under the imitation update rule.

Asynchronous imitation updates do not generally result in convergence to an equilibrium, but in Chapter 10, we have established convergence of such networks when all agents are *opponent*

coordinating agents, i.e., earn more when their neighbors play the same strategy that they do, than when the neighbors play other strategies. Formally, each diagonal entry of the payoff matrix of an opponent-coordinating agent i is greater than the off-diagonal in the same row:

$$\pi^i_{1,1} > \pi^i_{1,2}, \qquad \pi^i_{2,2} > \pi^i_{2,1}. \tag{11.7}$$

We now prove that such networks are also A-coordinating, leading to unique equilibrium convergence after offering incentives at equilibrium.

Proposition 11.3. *Every network game of opponent coordinating agents with the imitation update rule is A-coordinating.*

Proof. Consider two strategy vectors $y, z \in \{A, B\}^n$ satisfying (11.2), and let the network be at state y. First we look at the case when $f_i(y) = \{A\}$ for some agent $i \in \mathcal{V}$, implying that the highest-earning agent in the neighborhood $\mathcal{N}_i \cup \{i\}$ of agent i is an A-playing agent. Now, if the strategy of some of the B-playing agents are changed to A so that the network reaches z, then the payoff of no A-playing agent decreases and the payoff of no B-playing agent increases since all agents are opponent coordinating. Hence, the highest-earning agent in the neighborhood of agent i will still be an A-playing agent, yielding $f_i(z) = \{A\}$, resulting in (11.3). The case when $f_i(y) = \{A, B\}$ can be proven similarly. □

Therefore, A-monotonicity and unique equilibrium convergence follows directly from Theorem 11.1, as stated in the following.

Corollary 11.1. *Every network of opponent-coordinating agents is A-monotone and uniquely convergent.*

That is if in a network of opponent-coordinating agents that is at equilibrium at time 0, a (possibly unique) non-negative reward is offered to every agent whenever they play strategy A at some time $k \geq 0$, then no agent will switch from A to B at any time $k \geq 1$, and the whole network will reach a unique equilibrium state in finite time.

11.5 Control through payoff incentives

Using the unique equilibrium convergence properties of opponent-coordinating networks, we now investigate the efficient use of payoff incentives to drive a network of agents who update according to the asynchronous imitation rule from any undesired equilibrium toward a desired equilibrium in which all or at least more agents play strategy A.

11.5.1 Uniform reward control

Suppose that some central agency has the ability to offer a reward of $r_0 \geq 0$ to all agents whenever they play strategy A. The resulting payoff matrix is given by

$$\begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & \quad B \end{array} \\ \begin{pmatrix} a_i + r_0 & b_i + r_0 \\ c_i & d_i \end{pmatrix} \end{array}, \qquad a_i, b_i, c_i, d_i \in \mathbb{R},$$

for each agent $i \in \mathcal{V}$. The control objective in this case is the following. Let $\mathbf{B}$ denote the n-dimensional strategy vector in which each agent plays B.

Problem 11.1 (Uniform reward control)**.** *Given a network game $\Gamma = (\mathcal{G}, \pi, \mathcal{I})$ and initial strategies $x(0) \neq \mathbf{B}$, find the infimum reward r_0^* such that for every $r_0 > r_0^*$, $x_i(t)$ will reach A for every agent $i \in \mathcal{V}$.*

In networks of opponent-coordinating agents, it is relatively straightforward to compute the optimal value of r_0^* once we have established the properties in Section 11.3. First, we take advantage of the fact that the number of agents who converge to A is monotone in the value of r_0 due to the A-monotone property. Second, simulations of the network game are fast to compute due to the unique equilibrium property, established in Corollary 11.1. To understand the second implication, note that according to property (ii), all activation sequences will result in the same equilibrium; thus, we can choose a sequence consisting only of agents who will switch from B to A, which will have a maximum length of n before reaching equilibrium.

We begin by generating a set $\mathcal{R}$ containing all possible candidate infimum rewards. This set is generated by computing all possible payoff differences between agents playing B and agents playing A when they are neighbors or linked by another initially B-playing agent. Consider a network of opponent-coordinating agents that is at equilibrium at time zero. Let n_i^A denote the number of neighbors of agent i who initially play A. Since no agent switches from A to B, the possible payoffs of an agent i when playing A (resp. B) at any time step are contained in the sets

$$\Pi_i^A \triangleq \left\{a_i(n_i^A + \delta_i) + b_i(\deg_i - n_i^A - \delta_i) : \delta_i \in \Delta_i\right\}$$
$$\Pi_i^B \triangleq \left\{c_i(n_i^A + \delta_i) + d_i(\deg_i - n_i^A - \delta_i) : \delta_i \in \Delta_i\right\},$$

where

$$\Delta_i = \{0, 1, \ldots, \deg_i - n_i^A\}.$$

Now consider an agent s who initially plays B and has a neighbor j whose strategy was either initially A or became A at some other time. Since the payoff of agent j must be greater than that of all B-playing agents i in the neighborhood of agent s in order to cause agent s to switch to A, the reward given to agent j must be greater than $\frac{y_i^B - y_j^A}{\deg_j}$ for some $y_i^B \in \Pi_i^B$ and $y_j^A \in \Pi_j^A$. As we show in Proposition 11.4, this leads to the following set of all candidate infimum rewards

$$\mathcal{R} \triangleq \left\{ \frac{y_i^B - y_j^A}{\deg_j} \middle| y_i^B \in \Pi_i^B, y_j^A \in \Pi_j^A,\ j \in \mathcal{N}_s, i \in \mathcal{N}_s \cup \{s\}, x_i(0) = B, s \in \mathcal{V}, x_s(0) = B \right\} \cup \{0\}.$$

Proposition 11.4. *For a network of opponent-coordinating agents with initial strategies $x(0) \neq \mathbf{B}$, $r_0^* \in \mathcal{R}$.*

Proof. Should all agents' strategies be initially A, the result is trivial since $r_0^* = 0$. So consider the situation where at least one B-playing agent exists. We observe that the network will reach the state of all A after offering the reward r at time $k = 0$, if the following condition is satisfied: for every agent $s \in \mathcal{V}$ who initially plays B, there exists some time k^s such that $x_s(k^s) = B$ and

$x_s(k^s+1) = A$. Equivalently, for every initially B-playing agent $s \in \mathcal{V}$, there must exist some time k^s and A-playing neighbor $j \in \mathcal{N}_s$, $x_j(k^s) = A$, such that for all B-playing agents $i \in \mathcal{N}_s \cup \{s\}$, $x_i(k^s) = B$,

$$r \deg_j + u_j(k^s) > u_i(k^s).$$

Since $u_j(k^s) \in \Pi_j^A$, $u_i(k^s) \in \Pi_i^B$ and $x_i(0) = B$, the condition is satisfied if the following holds: for every initially B-playing agent $s \in \mathcal{V}$, there exists some time k^s and agent $j \in \mathcal{N}_s$, such that for all $i \in \mathcal{N}_s \cup \{s\}$, $x_i(0) = B$,

$$r \deg_j + y_j^A > y_i^B$$

for some $y_j^A \in \Pi_j^A$ and some $y_i^B \in \Pi_i^B$. Now since this is a sufficient condition for r to drive the network to the all-A state, we have that

$$r_0^* = \inf \left\{ r \middle| r > \frac{y_i^B - y_j^A}{\deg_j}, y_i^B \in \Pi_i^B, y_j^A \in \Pi_j^A,\ j \in \mathcal{N}_s, i \in \mathcal{N}_s \cup \{s\}, x_i(0) = B, s \in \mathcal{V}, x_s(0) = B \right\},$$

implying that

$$\begin{aligned} r_0^* \in & \left\{ r \middle| r = \frac{y_i^B - y_j^A}{\deg_j}, y_i^B \in \Pi_i^B, y_j^A \in \Pi_j^A,\ j \in \mathcal{N}_s, i \in \mathcal{N}_s \cup \{s\}, x_i(0) = B, s \in \mathcal{V}, x_s(0) = B \right\} \\ & = \mathcal{R} - \{0\}. \end{aligned}$$

By summarizing this case and the case when $r_0^* = 0$, we arrive at the proof. □

Next we sort the elements of $\mathcal{R}$ from low to high and denote this vector by $v^{\mathcal{R}}$. Algorithm 3 performs a binary search over $v^{\mathcal{R}}$ to find the infimum reward such that all agents in the network will eventually play A. Denote by $\mathbf{1}$ the n-dimensional vector containing all ones, and let $\deg_{\max}$ denote the maximum degree of the network. In what follows, we also denote by $\bar{x}$ the unique equilibrium resulting from a particular set of incentives being offered to a network of A-coordinating agents starting from x.

$i^- := 1$
$i^+ := |\mathcal{R}|$
while $i^+ - i^- > 1$ **do**
  $r_0^* := v_j^{\mathcal{R}}$, where $j := \lceil \frac{i^- + i^+}{2} \rceil$
  $\Gamma' := (\mathcal{G}, \pi', \mathcal{I})$
  Simulate Γ' from $x(0)$ until equilibrium $\bar{x}$
  if $\bar{x}_i = A$ *for all* $i \in \mathcal{V}$ **then**
    $i^+ := j$
  else
    $i^- := j$
  end
end

Algorithm 3: Binary search over candidate rewards to find the value of r_0^* that solves Problem 11.1 for networks of opponent-coordinating agents.

Proposition 11.5. *Algorithm 3 computes the reward r_0^* that solves Problem 11.1 for networks of opponent-coordinating agents.*

Proof. Since $r_0^* \in \mathcal{R}$ due to Proposition 11.4, the minimum $r_0 \in \mathcal{R}$ which results in all agents eventually playing A is r_0^*. Due to Theorem 11.1, we know that if a given r_0 results in all agents switching to A for one activation sequence, then it does so for every activation sequence. Therefore, we can test any given r_0 by activating only those agents who have higher earning neighbors who are playing a different strategy. Since agents in opponent-coordinating networks can only switch from B to A after a decrease in thresholds, such a simulation requires no more than n activations. We know from Corollary 11.1 that offering incentives to a network of opponent-coordinating agents will cause no agent to switch from A to B, which means we can perform a binary search on the ordered list $v^{\mathcal{R}}$. □

11.5.2 Targeted reward control

Suppose that rather than offering a uniform incentive to all agents who play strategy A, one has the ability to offer a different reward to each agent. By targeting the most influential agents in the network, it may be possible to achieve the desired outcome at much lower cost than with uniform rewards, but which agents should be targeted and how much reward should be offered to each of these agents?

Let $r := (r_1, \dots, r_n)^T$ denote the vector of rewards offered to each agent, where $r_i \in \mathbb{R}_{\geq 0}$ is the reward offered to agent i, resulting in the following payoff matrix for each agent $i \in \mathcal{V}$:

$$\hat{\pi} := \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} a_i + r_i & b_i + r_i \\ c_i & d_i \end{pmatrix} \end{array}, \quad a_i, b_i, c_i, d_i \in \mathbb{R}.$$

The targeted control objective is the following.

Problem 11.2 (Targeted reward control)**.** *Given a network game $\Gamma = (\mathcal{G}, \pi, \mathcal{I})$ and initial strategies $x(0)$, find the targeted reward vector r^* that minimizes $\sum_{i \in \mathcal{V}} r_i^*$ such that if $r_i > r_i^*$ for each i, then $x_i(k)$ will reach A for every agent $i \in \mathcal{V}$.*

Towards a solution to this problem, we first observe that for a network at some equilibrium state $\bar{x}$, the only way to get imitating agents to switch from B to A through positive rewards is to offer those rewards to agents who are sure to play A at some time and who have at least one neighbor playing B. For such an agent, the infimum reward such that at least one B-playing neighbor will switch to A is

$$\check{r}_i = \max_{j \in \mathcal{N}_i^B} \bar{y}_j - \bar{y}_i, \tag{11.8}$$

where $\bar{y}_i$ and $\bar{y}_j$ denote the payoffs of agents i and j at the equilibirum state $\bar{x}$, and $\mathcal{N}_i^B := \{j \in \mathcal{N}_i \cup \{i\} : \bar{x}_j = B\}$ denotes the self-inclusive set of neighbors of agent i who are playing B. Due to Corollary 11.1, offering this reward to agent i will result in unique new equilibrium regardless of the sequence in which agents activate. As a result, we can repeatedly use (11.10) to compute infimum rewards starting from each new equilibrium. Indeed, any algorithm which iteratively

offers rewards in this manner will produce a reward vector that achieves uniform convergence of the network to strategy A. A generic version of such an algorithm is described below, in which the key step is the choice of the agent j at each iteration, and ϵ denotes an arbitrarily small positive constant.

Initialize $\bar{x}_i = x_i(0)$ and $r_i = 0$ for each $i \in \mathcal{V}$
while $\exists i \in \mathcal{V} : \bar{x}_i \neq A$ **do**
  $\mathcal{A}^B := \{i \in \mathcal{V} : \bar{x}_i = A \wedge [\exists j \in \mathcal{N}_i : \bar{x}_j = B]\}$
  Choose an agent $j^* \in \mathcal{A}^B$
  Let $r_{j^*} := r_{j^*} + \check{r}_{j^*} + \epsilon$
  Let $\Gamma = (\mathcal{G}, \hat{\pi}, \mathcal{I})$
  $\bar{x} :=$ next equilibrium of Γ starting from $\bar{x}$
end

Algorithm 4: Generic iterative algorithm for computing a reward vector such that all agents in the network will play strategy A.

It is possible to find the exact solution r^* to Problem 11.2 by performing an exhaustive search that considers every possible sequence of agents in the sets $\mathcal{A}^B$. However, the computational complexity of such an algorithm prohibits its use on large networks. In Section 11.6, we explore the use of various heuristics for choosing an agent to target at each iteration, including random selection, highest degree, and highest payoff. Next, we propose a slightly more advanced heuristic for incentive targeting inspired by a similar approach to controlling best-response networks in Chapter 9.

Consider a network of opponent-coordinating agents, which is at some equilibrium state $\bar{x}$. In order to identify which agents should be offered incentives, we propose a simple potential function

$$\Phi(x) = \sum_{i=1}^{n} n_i^A, \tag{11.9}$$

where n_i^A denotes the number of neighbors of agent i who play strategy A in the state x. This function has a unique maximum, which occurs when all agents play A, and increases whenever an agent switches from B to A. Problem 11.2 translates to finding the infimum reward vector that maximizes this potential function. Therefore we propose a type of greedy algorithm which iteratively chooses the agent who maximizes the ratio of the change in potential to the reward required to achieve the corresponding change. Let $\bar{x}'$ denote the equilibrium resulting from offering the reward $\check{r}_j$ to agent j. Then we define the iterative potential to reward algorithm (IPRO) as Algorithm 4 in which the targeted agent is selected as follows.

$$j^* = \arg\max_{j \in \mathcal{A}^B} \frac{\Delta\Phi(\bar{x})^\alpha}{\check{r}_j^\beta}, \tag{11.10}$$

where $\Delta\Phi(\bar{x}) := \Phi(\bar{x}') - \Phi(\bar{x})$.

11.5.3 Budgeted targeted reward control

In this section, we suppose that there is a limited budget from which to offer rewards and pose the following dual problem to the one in the previous section.

Problem 11.3 (Budgeted targeted reward control)**.** *Given a network game* $\Gamma = (\mathcal{G}, \pi, \mathcal{I})$*, initial strategy state* $x(0)$*, and budget constraint* $\sum_{i\in\mathcal{V}} r_i < \rho$*, find the reward vector* r^* *that maximizes the number of agents in the network who reach* A*.*

Algorithm 5 slightly modifies Algorithm 4 to approximate the solution to Problem 11.3. The only difference is that the algorithm will now terminate if no more agents can be offered a reward without violating the budget constraint ρ.

Initialize $\bar{x}_i = x_i(0)$ and $r_i = 0$ for each $i \in \mathcal{V}$
while $\exists i \in \mathcal{V} : \bar{x}_i \neq A$ *and* $\sum_{i\in\mathcal{V}} r_i < \rho$ **do**
  $\mathcal{A}^B := \{i \in \mathcal{V} : \bar{x}_i = A \wedge [\exists j \in \mathcal{N}_i : \bar{x}_j = B]\} \wedge \check{r}_i \leq \rho - \sum_{i\in\mathcal{V}} r_i\}$
  Choose an agent $j \in \mathcal{A}^B$
  Let $r_j := r_j + \check{r}_j + \epsilon$
  Let $\Gamma = (\mathcal{G}, \hat{\pi}, \mathcal{I})$
  $\bar{x} :=$ next equilibrium of Γ starting from $\bar{x}$
end

Algorithm 5: Generic iterative algorithm for computing a reward vector to approximate the maximum number of agents who will play strategy A.

11.6 Simulations

Here we compare the performance of the IPRO algorithm to some alternative approaches for controlling networks of agents with imitative dynamics. Each of these methods is applied iteratively, targeting agents with payoff rewards until either the control objective is achieved or the budget limit is reached. Short descriptions of each algorithm under consideration are provided below.

- **Iterative Random (rand):** target random agents in the network
- **Iterative Degree-Based (deg):** target agents with maximum degree
- **Iterative Maximum Earning (IME):** target A-playing agents earning the highest payoffs while having at least one neighbor playing B
- **Iterative Potential Optimization (IPO):** target agents resulting in the maximum increase of the potential function ($\alpha = 1$, $\beta = 0$)
- **Iterative Reward Optimization (IRO):** target agents requiring minimum reward ($\alpha = 0$, $\beta = 1$)

- **Iterative Potential-to-Reward Optimization (IPRO):** target agents maximizing the potential-change-to-reward ratio ($\alpha = 1$, $\beta = 1$)
- **Optimal:** perform exhaustive search to find optimal solution (only practical for small networks)

For each set of simulations, we generate geometric random networks by randomly distributing n agents in the unit square and connecting all pairs of agents who lie within a distance R of each other.

Heterogeneous payoffs for the agents are generated as follows:$\pi_i = p_i I + v_i W_i$, where $p_i \geq 1$ denotes the coordination level, $v_i \in [0, 1]$ denotes the payoff variance, and W_i is a 2×2 matrix whose elements are drawn independently at random from a uniform distribution on the interval $[0, 1]$. Also, the W_i matrices are independent across all agents. Next, we introduce four simulation studies and provide graphical results, which are also summarized in Table 11.1.

11.6.1 Uniform vs. targeted reward control

First, we investigate the difference between uniform and targeted reward control to estimate the expected cost savings when individual agents can be targeted for rewards rather than offering a uniform reward to all agents. Figure 11.1 shows that targeted reward control offers a large cost savings over uniform rewards, but also that the savings decreases as the networks get larger. This differs from the results of the same comparison on best-response networks, in which the opposite effect was observed in Chapter 9.

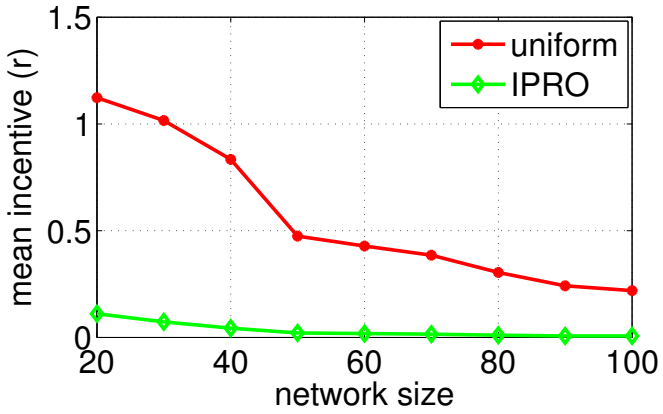


Figure 11.1: Comparison of uniform and targeted reward control on geometric random networks for a range of sizes. For each size tested, 100 random networks were generated using a connection radius $R = \sqrt{(1 + \deg_{\text{exp}})/\pi n}$, corresponding to a mean node degree of approximately $\deg_{\text{exp}} = 4$. The parameter values used to generate the payoffs were $p_i = 1$ and $v_i = \frac{1}{2}$.

11.6.2 Targeted-reward control: network size

Next, we compare algorithm performance for various sizes of networks of opponent-coordinating agents, using the same network setup as the previous section. Fig. 11.2 shows that the IPRO and

degree-based algorithms perform the best of the tested algorithms across all network sizes.

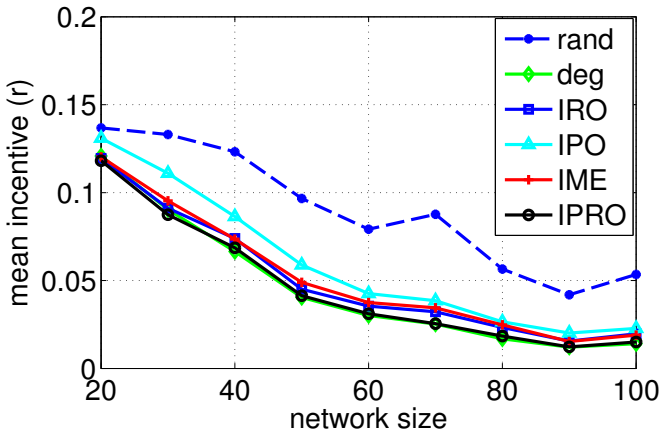


Figure 11.2: Comparison of algorithms for different sizes of networks. The connection radius, threshold distribution, and payoffs are generated exactly as in the simulations for Fig. 11.1.

11.6.3 Targeted-reward control: network connectivity

In this section, we investigate the effect of network connectivity on the total reward required to achieve consensus in strategy A. We consider geometric random networks of 20 agents, which is small enough that we can compute the optimal solution using an exhaustive search algorithm and compare this with the proposed algorithm. Fig. 11.3 shows that there is a sharp decrease in the mean incentive required as the networks become more densely connected. All of the algorithms except for random and IPO yielded near-optimal results in these tests, with IPRO performing the best.

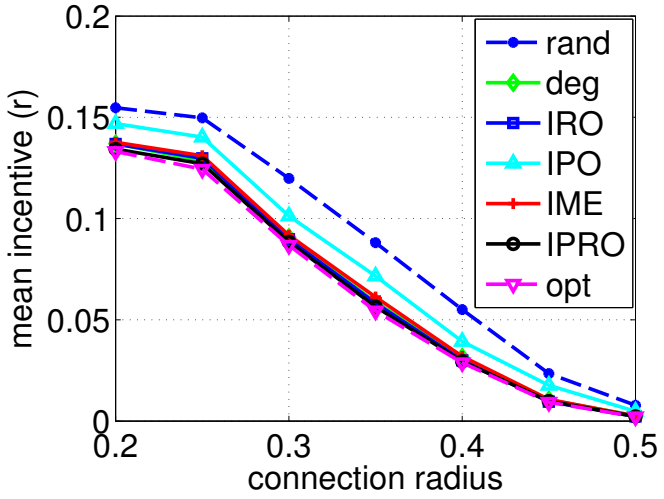


Figure 11.3: Comparison of algorithms on 500 sparsely to densely connected 20-node geometric random networks.

11.6.4 Targeted-reward control: payoff variance

In the next set of simulations, we vary the parameter v_i to understand how the algorithms perform for varying degrees of heterogeneity of the agents. Fig. 11.4 shows that the IPRO algorithm performs the best of the algorithms regardless of the degree of homogeneity or heterogeneity of the agents.

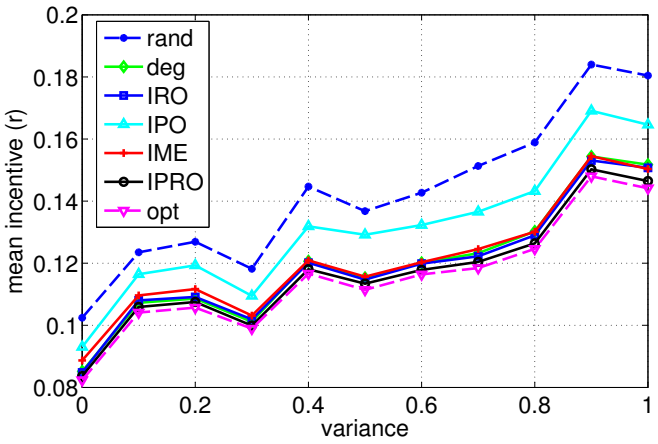


Figure 11.4: Comparison of algorithms for different amounts of payoff variance v_i. 500 20-node networks are tested for each value of v_i.

Table 11.1: Simulation Results: Mean Incentives

Algorithm	Fig. 2	Fig. 3	Fig. 4
rand	.090	.086	.143
deg	**.046**	.066	.120
IRO	.051	.065	.119
IPO	.060	.075	.131
IME	.052	.067	.121
IPRO	**.046**	**.064**	**.117**
opt	–	.063	.116

11.7 Concluding remarks

We have revealed three properties of asynchronous *A-coordinating* network games under any update rule after rewards are offered to agents when the network is at equilibrium: (i) no agent will switch from A to B; (ii) switches occur independent of the sequence in which agents activate; (iii) the

network will converge to a unique equilibrium. This predictability after offering rewards facilitates the design of efficient and in some cases optimal control protocols. We have further shown that a subset of networks in which agents asynchronously imitate their highest earning neighbor, i.e., networks of opponent-coordinating agents, are indeed A-coordinating, and therefore satisfy the above three properties. Based on this result, we proposed protocols for three control problems that apply to this class of networks: uniform reward control, targeted reward control, and budgeted targeted reward control. In particular, our proposed IPRO algorithm, which iteratively chooses agents who maximize the ratio of change in potential to offered reward, performs near-optimal in several different cases and outperforms those based on other heuristics such as maximum payoff-earning or minimum required-reward.

Chapter 12

Concluding remarks

12.1 Conclusion

12.1.1 Part I: infinite well-mixed populations

In Chapter 2, we have focused on continuously differentiable planar vector fields and revisited the celebrated Poincaré Bendixson Theorem. We have tightened the theorem for a positively invariant, simply-connected compact set $\mathcal{M}$ by further characterizing for any point $p \in \mathcal{M}$, the composition of the limit sets $\omega(p)$ and $\alpha(p)$ after counting separately the fixed points on $\mathcal{M}$'s boundary and interior. Compared to the classical form of Poincaré-Bendixson theorem, what we have further clarified is the role the interior fixed points of $\mathcal{M}$ play to influence the topological structure of the limit sets. The results can be used to reduce the number of possible limit sets of p, which we have illustrated by carrying out global convergence analysis for planar replicator dynamics in three examples.

In Chapter 3, we have studied n-dimensional, $n \in \mathbb{N}$, continuously differentiable vector fields in $\mathbb{R}^n$, possessing an invariant compact curve Γ or more generally an invariant compact manifold $\mathcal{M}$, to which the trajectory through some point $p \in \mathbb{R}^n$ converges. For the case of the curve, we have shown that the only possibilities for $\omega(p)$ are a single fixed point, a continuum of fixed points or the union of compactly oriented orbits of Γ and possibly some attached continuum of fixed points. For the case of a manifold, we have shown that the area between each attracting (resp. repelling) set of the vector field restricted to $\mathcal{M}$ and the boundary of its region of attraction (resp. repulsion) does not intersect the limit set, even when the attracting and repelling sets partially coincide with the boundary of $\mathcal{M}$. The results shed light on the qualitative property of a vector field near a compact manifold, which we have illustrated by several 4-dimensional replicator dynamics.

In Chapter 4 we have focused on some 4-dimensional replicator dynamics in a population of individuals playing a repeated snowdrift game with four typical strategies: ALLC, TFT, STFT and ALLD. After parameterizing the corresponding payoff matrix, we have demonstrated how the asymptotic behavior of the resulting 3-dimensional system evolves as the mutual cooperation payoff changes, and in particular shown that for the full range of payoffs, every trajectory of the system converges to an equilibrium point. The convergence results help to better understand how repetition can promote cooperation in populations playing snowdrift games. The results also enables us to compare the performance of the four strategies, indicating that an ALLC-player,

although perceived as the one that can be easily taken advantage of in snowdrift games, has certain endurance in the long run. Finally, the parametric framework makes it possible to actually control the final population shares by tuning the payoffs of the base game.

In Chapter 5 we have proceeded to arbitrary-dimensional replicator dynamics in a population of individuals playing 2×2 games with preplay communication. We have studied the evolution of the population shares of four types of individuals in this setup: homophilic cooperators, heterophilic cooperators, pure cooperators and pure defectors. Besides having revealed the innate properties of homophilic and heterophilic cooperators, we have addressed previous research on prisoner's dilemma games with preplay communication by proving that the frequencies of all but pure defectors converge to zero in populations containing all four types. This urges studying cheap-talk games under other reproduction dynamics or when the population is structured.

12.1.2 Part II: finite well-mixed populations

In Chapter 6, we have studied how sustainable cooperation might emerge among self-interested, anti-coordinating individuals in well-mixed populations governed by the best-response update rule. We have shown that one can always identify one type of individuals as a benchmark such that after a sufficiently long but finite time, individuals more cooperative compared to the benchmark almost surely always cooperate while those less cooperative compared to the benchmark almost surely defect. Such insight provides theoretical explanation for some complex behavior recently reported in simulation studies that highlight the puzzling effect of individuals' heterogeneity on collective decision-making dynamics. As a second contribution, we have demonstrated how to control the total number of cooperators by changing the payoff matrices of the agents, resulting in changes in their types. In particular, for a given population, we have found the minimum number of changes in the types required to reach a desired number of cooperators in the final state. The results indicate that the minimum number does not equal the difference of the current number of cooperators from the desired one. This highlights the side-effects the level-off phenomenon can cause in such models.

In Chapter 7, we have focused on perception differences among the individuals of a population using the linear threshold model. We have shown that despite the simplicity of the decision-making model, the population dynamics may exhibit several possible long-run behavior for the same initial condition. We have found all those possible equilibria that the dynamics may reach from a given initial condition, and shown that in contrast to the case of homogeneous populations, the heterogeneity in agents' thresholds give rise to several equilibria where both A-playing and B-playing agents coexist. Such results highlight the crucial role of the heterogeneity of the population in linear threshold models for the coexistence of individuals playing competing strategies.

12.1.3 Part III: finite networked populations

In Chapter 8, we have focused on networks of agents governed by asynchronous best-response dynamics in 2×2 games, or equivalently, a threshold-based model, and shown that every such network will reach an equilibrium in finite time, even if each agent has a different threshold, provided that all agents are either *anti-coordinating*, i.e., choose a strategy only if a small enough portion of neighbors are also using that strategy, or *coordinating*, i.e., choose a strategy only

if a large enough portion of neighbors are also using that strategy. These results reveal that irregular network topology, population heterogeneity and partial synchrony are not sufficient to cause non-convergence of best-response dynamics in two-strategy games; rather, other factors such as randomness, or the occasional use of non-best-response strategies must play a role.

In Chapter 9, by building upon the convergence results on networks of interacting agents modeled in Chapter 8, we have proposed control algorithms to efficiently drive such networks to a desired equilibrium state by offering payoff incentives or rewards for using a particular strategy, either uniformly or targeted to individuals. Simulations have shown that our proposed algorithms can be used to compute near-optimal targeted incentives for a wide range of networks and payoff distributions in coordination games and can also be effective for anti-coordination games.

In Chapter 10, we have focused on populations governed by the imitation dynamics and studied the conditions that force the population to converge to a state in which the agents are satisfied with their decisions. We have shown that every network in which agents imitate the best performing strategy in their neighborhood will reach an equilibrium in finite time, provided that all agents are opponent-coordinating. Comparing to the best-response case, this implies that networks of imitating agents are less likely to settle on an equilibrium state.

In Chapter 11, we have considered how to control the network modeled in Chapter 10, by offering payoff incentives to individuals. In particular, we have shown that if the network is A-coordinating, then providing incentives to the network in an equilibrium will result in convergence to a *unique* equilibrium. This is a generalization of the result in Chapter 9 and allows the computation of optimal uniform incentives using a binary search algorithm, and enables us to design near-optimal targeted and budgeted targeted control algorithms.

12.2 Recommendations for future research

12.2.1 Part I: infinite well-mixed populations

First we mention possible future work for the mathematical results achieved in Chapters 2 and 3, and then we discuss the evolutionary game theoretical results developed in Chapters 4 and 5.

The current state-of-the-art stays far away from classifying the possible limit sets of an arbitrary point in a positively invariant compact region in spaces with dimensions greater than two. It would, therefore, be of great interest to develop Poincaré-Bendixson-like theorems in higher dimensions, even for points in compact regions with particular structures. The extension of the results in Chapter 2 can be helpful in this process. For example, one can investigate Theorem 2.4 for positively invariant simply-connected compact regions in $\mathbb{R}^3$, whose interior are empty of fixed points. The results then provide some intuition to extend Poincareé-Bendixson theorem to regions in $\mathbb{R}^3$ possessing one or more interior fixed points.

Another approach for the classification of the possible limit sets of an arbitrary point in spaces with dimensions greater than two is to focus on the case when the trajectory through the point is known to converge to some compact set and determine where in that set it may actually converge to. This has been the main topic of Chapter 3 where the compact set is simply a curve or an arbitrary-dimensional manifold. Although the results of Section 3.3 have been already generalized in Section 3.5 from a curve to a manifold, they only imply that the areas between the (semi-

)attracting and (semi-)repelling sets and the boundary of their attraction and repulsion regions do not intersect with the limit set of the arbitrary point. This is despite the fact that the non-fixed orbits connected to the fixed points on the open curve in Chapter 3.3 never intersect with the limit set, even when the fixed points are not an attractor or repellor. So one can investigate that besides the attracting and repelling sets, which other types of sets in the manifold have the property that the area connected to them do not intersect with the limit set. Another possible future work would be to extend the results on hyperbolic manifolds in Section 3.5 to the case when the manifolds are not necessarily diffeomorphic to a plane.

To further reveal the performance of the conditional strategies in Chapter 4, one can try to obtain similar convergence results for the replicator dynamics with the same payoff matrix but under different base games such as the prisoner's dilemma or the coordination game. Another possibility is to add a fifth strategy such as the win-stay loose-shift [115] or the outstanding zero-determinant strategy [133]. Investigating the replicator dynamics in population mixtures of even three or four conditional strategies containing the zero-determinant strategy reveals the range of payoffs under which this reactive strategy outperforms the others.

As with Chapter 5, now that the innate properties of homophilic and heterophilic cooperators have been revealed, it is of great interest to investigate their performance against pure cooperators and defectors in games other than the prisoner's dilemma. For example, it remains an open question whether homophily and heterophily can help the individuals of a population to coordinate on a particular strategy in games such as the coordination game, or which types of players survive in the long run under the snowdrift game.

12.2.2 Part II: finite well-mixed populations

There are two main topics that deserve great attention in this part. First is studying the asymptotic behavior of the best-response dynamics in well-mixed populations containing both coordinating and anti-coordinating agents. The corresponding results will shed light on the long run behavior of the dynamics when the mixed-population has a more complicated network structure. Second is studying the well-mixed populations introduced in Chapters 6 and 7 under the imitation update rule, which we have started to work on. In contrast to the case with the best-response update rule, populations of imitating individuals exhibit non-convergence behavior in simulations, making it quite challenging to characterize the asymptotic behavior of the dynamics. Then one can try to control the population using a similar approach to that in Chapter 6.

12.2.3 Part III: finite networked populations

We have shown in Chapter 8 that every network of all coordinating or all anti-coordinating agents converges to an equilibrium state under the best-response update rule. However, it remains an open problem that for which combination of networks and agents' thresholds, a population of both coordinating and anti-coordinating agents converges to an equilibrium state. The same holds for the imitation update rule in Chapter 10, namely for which combination of networks and agents' payoff matrices, a population containing both opponent-coordinating and non-opponent-coordinating agents converges to an equilibrium. Another interesting problem is to investigate the asymptotic behavior of a population containing agents who update their strategies according to

the best-response update rule and agents who update according to the imitation update rule. The solutions to such problems help to find the cause of non-convergence often happening in social networks.

On the other hand, for control of networks governed by the best-response and imitation update rules, an enticing subject is to design efficient control algorithms for general A-coordinating network games. The algorithms will be perhaps extensions of those presented in Chapters 9 and 11. The results then can be applied to networks with other update rules, such as the death-birth [120].

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Summary

Networks of decision-making individuals with simple dynamics may give rise to complex and seemingly unpredictable collective behaviors which may have negative consequences such as traffic jams and market crashes or positive outcomes such as volunteer disaster relief and free-market stabilization. Usually on one hand, there is a group task requiring the individuals to cooperate to optimize the collective performances, and on the other, each individual is self-interested and may prefer to prioritize her own interest over the group task, resulting in a social dilemma. Typical research questions in this topic therefore focus on (i) how to predict the long run behavior of the networks and find factors causing non-converging fluctuations in the actions of the individuals; (ii) how to build a model to describe the dynamics as cooperation evolves or is promoted among selfish individuals; and (iii) how to control the number of individuals taking a particular action in a network. We tackle these problems by modeling networks of decision-making individuals using evolutionary matrix-game dynamics, performing convergence analysis to understand the asymptotic behavior of the dynamics, and designing control algorithms to lead the individuals to a desired action. This is done in three parts, as explained in the following.

In Part I, we focus on infinite, well-mixed populations, leading to continuous dynamics, the most well-known of which are the replicator dynamics. We start with developing mathematical results that prove to be useful in analyzing the replicator dynamics as well as other continuously differentiable vector fields. We revisit the celebrated Poincaré Bendixson Theorem and tighten it for a positively invariant, simply-connected planar compact set $\mathcal{M}$ by further characterizing for any point $p \in \mathcal{M}$, the composition of the limit sets $\omega(p)$ and $\alpha(p)$ after counting separately the fixed points on $\mathcal{M}$'s boundary and interior. We reveal the role that the interior fixed points of $\mathcal{M}$ play to influence the topological structure of the limit sets, and show, compared to classical results, how to reduce the number of possible limit sets of p. Then we proceed to continuously differentiable vector fields in $\mathbb{R}^n$, $n \in \mathbb{N}$, possessing an invariant compact curve Γ, to which the trajectory through some point $p \in \mathbb{R}^n$ converges, and determine the possible limit sets of p. We extend the results to when instead of the curve, the vector field possesses an arbitrary-dimensional invariant compact manifold $\mathcal{M}$. The results shed light on the qualitative property of a vector field near a compact manifold. Then we study two promising mechanisms for the promotion of cooperation: direct reciprocity, and preplay communication. For the first, we investigate the asymptotic behavior of a population of individuals playing repeated snowdrift games with four conditional strategies under the replicator dynamics. We compare the performance of the strategies and show how the inclusion of reactive strategies increases the share of cooperators of the overall population compared to when the game is not repeated. For the second, we study replicator dynamics for a population consisting of four types of individuals playing 2×2 games with preplay communication: homophilic cooperators,

heterophilic cooperators, pure cooperators and pure defectors. We reveal the innate properties of homophilic and heterophilic cooperators, and show that under the prisoner's dilemma, pure cooperators survive in face of homophilic cooperators but vanish in the presence of heterophilic cooperators. We also prove that the frequencies of all but pure defectors converge to zero in populations containing all four types, which urges studying preplay communication under other reproduction dynamics.

In Part II, we focus on finite well-mixed populations, leading to discrete dynamics, the most well-known of which are the best-response dynamics. We consider heterogeneous populations where each individual is associated with a possibly unique payoff matrix, chooses between two options A and B, or cooperation and defection, and updates her choice asynchronously according to the best-response update rule. We start with how sustainable cooperation might emerge among self-interested, anti-coordinating individuals in well-mixed populations. We provide theoretical explanations for some complex behavior recently reported in simulation studies that highlight the puzzling effect of individuals' heterogeneity on collective decision-making dynamics. We then demonstrate how to control the total number of cooperators in such heterogeneous populations, by influencing the individuals' payoffs gained from the games over time. In particular, we find the minimum number of individuals whose payoff matrices must be changed for the population to reach a desired number of cooperators in the long run. Finally, we focus on perception differences among the individuals of a population using the linear threshold model. We show that in contrast to the case of homogeneous populations, the heterogeneity in individuals' thresholds gives rise to several equilibria where both A-playing and B-playing individuals coexist. This highlights the heterogeneity of the population for the coexistence of individuals playing competing strategies.

In Part III, we focus on finite structured populations, leading to again discrete dynamics, two well-known of which are the best-response and imitation dynamics. We first focus on networks of individuals governed by asynchronous best-response dynamics in 2×2 games, or equivalently, a threshold-based model, and show that every such network will reach an equilibrium in finite time, even if each individual has a different threshold, provided that all individuals are either anti-coordinating, or coordinating. The results reveal that irregular network topology, population heterogeneity and partial synchrony are not sufficient to cause non-convergence of best-response dynamics in two-strategy games; rather, other factors such as randomness, or the occasional use of non-best-response strategies must play a role. Building upon these convergence results, we then propose control algorithms to efficiently drive such networks to a desired equilibrium state by offering payoff incentives or rewards for using a particular strategy, either uniformly or targeted to individuals. Then we focus on populations governed by imitation dynamics and show that every network in which individuals imitate the best performing strategy in their neighborhood will reach an equilibrium in finite time, provided that all individuals are opponent-coordinating, i.e., earn a higher payoff if their opponent plays the same strategy as they do. Comparing to the best-response case, this implies that networks of imitating individuals are less likely to settle on an equilibrium state. Then we consider how to control networks of imitating individuals by offering payoff incentives. By generalizing our control results on the best-response dynamics to a broad class of network games, we design a binary search algorithm to compute the optimal uniform incentive, and design near-optimal targeted and budgeted targeted control algorithms.

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